# Part III Stochastic Calculus
# Based on lectures by J. Miller

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## Contents



Lecture 1

# 1 Motivation

This course is about developing a theory of calculus which is applicable to continuous time stochastic processes, e.g. Brownian motion. Why do we need a special theory?

Brownian motion is **not differentiable**.

| **Ordinary calculus** | **Stochastic calculus** |
|---|---|
| Integral | Itô (stochastic) integral |
| Derivative | Itô (stochastic) derivative |
| ODEs | SDEs |

**Example:** Suppose that we have a gambler who repeatedly tosses a fair coin, betting \$1 on getting a heads for each toss. Let

$$\xi_k = \begin{cases} +1, & \text{heads on } k\text{th toss} \\ -1, & \text{otherwise}. \end{cases}$$

That is, the $(\xi_k)$ are i.i.d. Bernoulli($\pm 1$). Let

$$X_n = \sum_{k=1}^{n} \xi_k$$

be the net winnings of the gambler. Note that $(X_n)$ is a simple random walk and $X_0 = 0$, hence is a martingale (MG) w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Suppose that at the $n$th toss, bet $h_k$ on heads. Then

$$(H \cdot X)_n = \sum_{k=1}^{n} h_k (X_k - X_{k-1}) .$$

We interpret $(H \cdot X)_n$ as the gains process from a self-financing strategy $H$ which gives the net winnings after $n$ tosses. Assume that $(H_n)$ is a deterministic sequence.

<u>Claim:</u> $(H \cdot X)_n$ is an $\mathcal{F}_n$-martingale.

(a) $H_k$ is integrable ✓

(b) $H_k$ is adapted ✓

(c) $\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \mid \mathcal{F}_n] = H_{n+1} \cdot \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 0.$

More generally, the same is true if we take $H_{n+1}$ to be $\mathcal{F}_n$-measurable *(and integrable)*. This is called a **previsible process**. As before, $(H \cdot X)$ gives the net winnings of the gambler. This is called a **martingale transform**.

<u>Goal for first part of the course:</u> Extend this reasoning to define the *stochastic integral*

$$(H \cdot X)_t = \int_0^t H_s \, dX_s \qquad (\spadesuit)$$

where $H$ is previsible and $X$ is a continuous martingale (e.g., Brownian motion). Crucially, one cannot use the Lebesgue–Stieltjes integral to define ($\spadesuit$), since this requires $X$ to have finite variation, and the only continuous martingales with finite variation are *constant*, as we will show later in the course. Thus, our strategy to define the Itô Integral will be to set

$$(H \cdot X)_t := \lim_{\varepsilon \to 0} \sum_{k=1}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}(X_{k\varepsilon} - X_{(k-1)\varepsilon})$$

We need to be careful about the type of limit since $X$ in general will be rough (not differentiable), like Brownian motion. To get convergence, we need to take advantage of cancellations. For example, if $X$ is a Brownian motion and $H$ is a deterministic and continuous process. We have

$$\mathbb{E}\left[\left(\sum_{k\varepsilon \le t} H_{k\varepsilon}(X_{(k+1)\varepsilon} - X_{k\varepsilon})\right)^2\right] = \mathbb{E}\left[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2 + \sum_{k \ne \ell} H_{k\varepsilon} H_{\ell\varepsilon}(X_{(k+1)\varepsilon} - X_{k\varepsilon})(X_{(\ell+1)\varepsilon} - X_{\ell\varepsilon})\right]$$

$$= \mathbb{E}\left[\sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 (X_{(k+1)\varepsilon} - X_{k\varepsilon})^2\right] = \sum_{k=0}^{\lfloor t/\varepsilon \rfloor} H_{k\varepsilon}^2 \cdot \varepsilon \to \int_0^t H_s^2 \, ds \quad \text{as } \varepsilon \to 0.$$

Cancellations come from martingale orthogonality and are what make it possible to define the Itô integral.

Next, learn about properties of the integral:

- Stochastic analogue of the chain rule,
- Stochastic analogue of integration by parts.

Formulas look like those in regular calculus but with an extra term to reflect that $X$ is rough (quadratic variation).

$$Y_t = \int_0^t H_s \, dX_s \iff \mathrm{d}Y_t = H_t \, \mathrm{d}X_t \, .$$

Itô's formula will tell us how to write $df(Y_t)$ in terms of $\mathrm{d}Y_t$ for $f \in \mathcal{C}^2$. It has many applications, for example the Dubins–Schwarz theorem which states that *any continuous martingale is a time-change of Brownian motion*.

Next look at stochastic differential equations (SDEs), i.e.,

$$\mathrm{d}X_t = b(t, X_t) \, dt + \sigma(t, X_t) \, \mathrm{d}B_t,$$

where $b, \sigma$ are "nice" and $B$ is a Brownian motion. For $\sigma \equiv 0$, just an ODE. For $\sigma \not\equiv 0$, corresponds to adding noise which depends on time and the state of the system.

Last part of the course: diffusion processes and how they are related to SDEs, and how they can be used to solve PDEs involving $2^{\text{nd}}$ order elliptic equations (e.g., $\Delta$).

Next time we will start with some preliminaries (càdlàg processes, function of finite variation, integral against a function/process of finite variation).

## 2 Preliminaries

Lecture 2

## 2.1 Càdlàg processes, functions of finite variation

Recall that $\alpha : [0,\infty) \to \mathbb{R}$ is càdlàg if $\alpha$ is right-continuous and has left-hand limits:

$$\lim_{s\to t^+} \alpha(s) = \alpha(t), \quad \lim_{s\to t^-} \alpha(s) \text{ exists}.$$

Let $\alpha(x-), x \in [0,\infty)$ be right-hand limit, and set

$$\Delta\alpha(x) \coloneqq \alpha(x) - \alpha(x^-)\,, x \in [0,\infty)]\,.$$

Suppose that $\alpha$ is non-decreasing, càdlàg and $a(0) = 0$. Then there exists a unique Borel measure $\mathrm{d}\alpha$ on $([0,t],\mathcal{B})$ with

$$\mathrm{d}\alpha((s,t]) \coloneqq \alpha(t) - \alpha(s)\,, \text{ for all } 0 \le s < t\,.$$

For $f$ measurable and integrable, then the Lebesgue–Stieltjes integral of $f$ w.r.t. $\alpha$ is defined by

$$\int_{(0,t]} f(s)\,\mathrm{d}\alpha(s) \quad \forall t \ge 0.$$

Then, by dominated convergence $t \mapsto \int_{[0,t]} f(s)\,\mathrm{d}\alpha(s)$ is a right-continuous function. Moreover, if $f$ is continuous, then $t \mapsto \int_0^t f(s)\,\mathrm{d}\alpha(s)$ is continuous so we can write

$$\int_0^t f(s)\,\mathrm{d}\alpha(s) \coloneqq \int_{(0,t]} f(s)\,\mathrm{d}\alpha(s).$$

We want to integrate more general functions. Suppose that $\alpha^+, a^-$ are functions satisfying the same conditions as before, and set $a = a^+ - a^-$. Define $(f \cdot a)(t) = (f \cdot a^+)(t) - (f \cdot a^-)(t)$ for all $f$ measurable so that both terms on the RHS are finite. This class of functions (i.e., differences of càdlàg non-decreasing functions) coincides with the càdlàg functions with *finite variation*.

**Definition 2.1.** *Let $\alpha : [0,\infty) \to \mathbb{R}$ be càdlàg. For each $n \in \mathbb{N}, t \ge 0$, let*

$$v^n(t) \coloneqq \sum_{k=0}^{\lceil 2^n t\rceil - 1} \left|\alpha\left(\frac{(k+1)t}{2^n}\right) - \alpha\left(\frac{kt}{2^n}\right)\right|\,. \tag{$\circledast$}$$

*Then the limit $v(t)_t \coloneqq \lim_{n\to\infty} v^n(t)$ exists and is called the total variation of $\alpha$ on $[0,t]$. If $v(t)_t < \infty$ then we say that $\alpha$ has finite variation on $[0,t]$. If $v(t)_t < \infty$ for all $t \ge 0$, we say that $\alpha$ is a càdlàg function of finite variation.*

To see that $\lim v^n(t)$ exists, fix $t > 0$ and let

$$t_n^+ = 2^{-n}\lceil 2^n t\rceil \quad \text{so that } t_n^+ \ge t \ge t_n^- \quad \forall n$$

$$t_n^- = 2^{-n}\left(\lfloor 2^n t\rfloor - 1\right)$$

and

$$v^n(t) = \sum_{k=0}^{2^n t_n^- - 1} \left|a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})\right| + \left|a(t_n^+) - a(t_n^-)\right|.$$

The triangle inequality gives that the first term is non-decreasing in $n$. The càdlàg property implies that the second term converges to the jump $|\Delta a(t)|$, hence $v^n(t)$ converges as $n \to \infty$.

**Lemma 2.2.** *Let $a$ be a càdlàg function of finite variation. Then $v$ is also càdlàg of finite variation with $\Delta v(t) = |\Delta a(t)|$ for all $t \geq 0$ and $v$ is non-decreasing. In particular, if $a$ is continuous, then so is $v$.*

*Proof.* [✍] . ☐

**Proposition 2.3.** *A càdlàg function can be decomposed as a difference of two non-decreasing right-continuous functions if and only if it has finite variation.*

*Proof.* Assume that $a = a^+ - a^-$ are càdlàg, non-decreasing. NTS: $a$ has finite variation. Note,

$$|a(t) - a(s)| \leq \Big(a^+(t) - a^+(s)\Big) + \left(a^-(t) - a^-(s)\right) \qquad \forall\, 0 \leq s < t.$$

Plug this into ⊛ and use that the sum telescopes for monotone functions to get that

$$v^n(t) \leq \Big(a^+(t_n^+) - a^+(0)\Big) + \Big(a^-(t_n^+) - a^-(0)\Big).$$

Since $a^+, a^-$ are right-continuous,

$$\text{RHS} \overset{n\to\infty}{\longrightarrow} \Big(a^+(t) - a^+(0)\Big) + \left(a^-(t) - a^-(0)\right)$$

which gives that $a$ has finite variation, as desired.
Now the reverse direction. Assume that $v(t) < \infty$ for all $t > 0$. Set $a^+ = \frac{1}{2}(v+a)$, $a^- = \frac{1}{2}(v-a)$. Then $a = a^+ - a^-$ and $a^+, a^-$ are càdlàg since $v, a$ are càdlàg. NTS: $a^\pm$ are non-decreasing. Fix $0 \leq s < t$, define $t_n^{+/-}$ as before and $s_n^{+/-}$ analogously. Then:

$$\begin{aligned}
a^+(t) - a^+(s) &= \lim_{n\to\infty} \frac{1}{2}\Big(v^n(t) - v^n(s) + a(t_n^+) - a(s_n^+)\Big) \\
&= \lim_{n\to\infty} \frac{1}{2}\left[\sum_{k=2^n s_n^+}^{2^n t_n^- - 1} \left(|a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})| + a((k+1)\cdot 2^{-n}) - a(k\cdot 2^{-n})\right)\right. \\
&\qquad \left. + |a(t_n^+) - a(t_n^-)| + (a(t_n^+) - a(t_n^-))\right] \geq 0.
\end{aligned}$$

Same argument works for $a^-$. ☐

## 2.2 Random integrands

We now discuss integration against random functions of finite variation.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ filtered probability space. Recall that $X_t(\omega), t \in [0,\infty) \to \mathbb{R}$ is *adapted* to $(\mathcal{F}_t)_{t\geq 0}$ if $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. $X$ is *càdlàg* if $X(\omega, \cdot)$ is càdlàg for all $\omega \in \Omega$.

**Definition 2.4.** *Given a càdlàg, adapted process $A\colon \Omega \times [0,\infty) \to \mathbb{R}$, its total variation process $V\colon \Omega \times [0,\infty) \to \mathbb{R}$ is pathwise by setting $V(\omega, t)$ to be the total variation of $A(\omega, \cdot)$.*

**Lemma 2.5.** *If $A$ is càdlàg, adapted, and of finite variation then $V$ is càdlàg, adapted, and non-decreasing.*

*Proof.* Only NTS $V$ is adapted. For $t \geq 0$, $t_n^- = 2^{-n}(\lceil 2^n t \rceil - 1)$

$$\tilde{V}_t^n = \sum_{k=0}^{2^n t_n^- - 1} \left| A_{(k+1)\cdot 2^{-n}} - A_{k \cdot 2^{-n}} \right| ,$$

$\tilde{V}_t^n$ adapted for all $n$ since $t_n^- \leq t$.

$$V_t = \lim_{n\to\infty} \left( \tilde{V}_t^n + |\Delta A(t)| \right)$$

which shows that $V_t$ is $\mathcal{F}_t$-measurable. □

Lecture 3 We now seek a class of functions so that the integral is adapted.

Recall from the introduction that a discrete-time process $(H_n)_n$ is called previsible w.r.t. $(\mathcal{F}_n)$ if $H_{n+1}$ is measurable w.r.t. $\mathcal{F}_n$ for all $n$.

**Definition 2.6.** *The previsible $\sigma$-algebra $\mathcal{P}$ on $\Omega \times (0, \infty)$ is the $\sigma$-algebra which is generated by sets of the form $E \times (s, t]$ where $E \in \mathcal{F}_s$, $s < t$. A process $H : \Omega \times (0, \infty) \to \mathbb{R}$ is previsible if it is measurable with respect to $\mathcal{P}$.*

**Examples:**

1. $H(\omega, t) = Z(\omega) \cdot \mathbf{1}_{(t_1,t_2]}(t)$, $t_1 < t_2$, $Z$ is $\mathcal{F}_{t_1}$-measurable.
2. $H(\omega, t) = \sum_{k=0}^{n-1} Z_k(\omega) \cdot \mathbf{1}_{(t_k,t_{k+1}]}(t)$, for $0 = t_0 < \cdots < t_n$ and $Z_k$ is $\mathcal{F}_{t_k}$-measurable.

A *simple process*, will be important for the construction of the Itô integral.

**Remark.** *Simple processes are left-continuous and adapted. It turns out that $\mathcal{P}$ is the smallest $\sigma$-algebra on $\Omega \times (0, \infty)$ so that all left-continuous processes are measurable, . In general, càdlàg processes are not previsible, but their left-continuous modification is.*

**Proposition 2.7.** *Let $X$ be a càdlàg, adapted process and let $H_t = X_{t-}$, $t \geq 0$. Then $H$ is previsible.*

*Proof.* Since $X$ is càdlàg and adapted, it is clear that $H$ is left-continuous and adapted. For each $n$, set

$$H_t^n = \sum_{k=0}^{\infty} H_{k\cdot 2^{-n}} \cdot \mathbf{1}_{(k\cdot 2^{-n},(k+1)\cdot 2^{-n}]}(t)$$

Then $H^n$ is previsible for all $n$ and since $H$ is a left-continuous process,

$$\lim_{n\to\infty} H_t^n = H_t \quad \forall t \Rightarrow H \text{ is previsible as a limit of previsible processes.} \quad \square$$

□

**Remark.** *The proposition above implies that continuous, adapted processes are previsible.*

**Proposition 2.8.** *If $H$ is previsible, then $H_t$ is measurable w.r.t. $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : s < t)$, $\forall t \geq 0$.*

*Proof.* [✍] . $\square$

**Remark.** *The Poisson process $(N_t)$ is* not *previsible since $N_t$ is not $\mathcal{F}_{t-}$-measurable, where $(\mathcal{F}_t)$ is the natural filtration.*

Now we are going to see that integrating a previsible process $H$ against a càdlàg process with a.s. finite variation $A$ yields a well-defined and adapted càdlàg process of finite variation.

**Theorem 2.9.** *Let $A : \Omega \times (0, \infty) \to \mathbb{R}$ be a càdlàg process which is adapted and has finite variation $V$. Let $H$ be a previsible process with*

$$\int_{0<s\leq t} |H(\omega, s)| \, dV(s) < \infty \quad \forall t > 0, \, \omega \in \Omega. \tag{2.1}$$

*Then the process $H \cdot A : \Omega \times (0, \infty) \to \mathbb{R}$ given by*

$$(H \cdot A)(\omega, t) = \int_{(0,t]} H(\omega, s) \, dA(\omega, s), \tag{2.2}$$

*with*

$$(H \cdot A)(\omega, 0) = 0,$$

*is càdlàg, adapted and has finite variation.*

*Proof.* The integral in 2.2 is well-defined due to 2.1. Indeed, let $H^+ = \max(H, 0)$, $H^- = \max(-H, 0)$, and

$$A^{\pm} = \tfrac{1}{2}(V \pm A).$$

Then $H = H^+ - H^-$ and $A = A^+ - A^-$ and

$$H \cdot A = (H^+ - H^-) \cdot (A^+ - A^-) = H^+ \cdot A^+ + H^- \cdot A^- - H^+ \cdot A^- - H^- \cdot A^+.$$

All terms on RHS are finite by 2.1. Need to show:

1. $H \cdot 1$ is càdlàg,
2. adapted,
3. finite variation.

<u>Step 1.</u> Note that $\mathbf{1}_{(0,s]} \to \mathbf{1}_{(0,t]}$ as $s \downarrow t$ and $\mathbf{1}_{(0,s]} \to \mathbf{1}_{(0,t]}$ as $s \nearrow t$. By definition,

$$(H \cdot A)_t = \int H_s \cdot \mathbf{1}_{(s \in (0,t])} \, dA_s.$$

Hence,

$$\begin{aligned}(H \cdot A)_t &= \int H_s \cdot \lim_{r \downarrow t} \mathbf{1}_{(s \in (0,r])} \, dA_s \\ &\overset{\text{(DCT)}}{=} \lim_{r \downarrow t} \int H_s \cdot \mathbf{1}_{(s \in (0,r])} \, dA_s = \lim_{r \downarrow t} (H \cdot A)_r\end{aligned}$$

giving right-continuity. An analogous argument gives that $H \cdot A$ has left-limits, hence is càdlàg. Also,

$$\Delta(H \cdot A)_t = \int H_s \cdot \mathbf{1}_{(s=t)} \, dA_s = H_t \cdot \Delta A_t$$

Step 2. "Monotone class" style argument. Suppose

$$H = \mathbf{1}_{B \times (s,u]}, \quad B \in \mathcal{F}_s, \quad s < u.$$

Then

$$(H \cdot A)_t = \mathbf{1}_B \cdot (A_{t \wedge u} - A_{t \wedge s}), \text{ which is } \mathcal{F}_t\text{-measurable.}$$

Let $\mathcal{A} = \{Z \in \mathcal{P} : \mathbf{1}_Z \cdot A \text{ is adapted}\}$. Want to show: $\mathcal{A} = \mathcal{P}$. Let

$$\Pi = \{B \times (s,u] : B \in \mathcal{F}_s, \ s < u\}.$$

We have shown $\Pi \subseteq \mathcal{A}$, and know that $\Pi$ is a $\pi$-system generating $\mathcal{P}$. Not difficult to see that $\mathcal{A}$ is also a $d$-system, and by Dynkin's lemma we deduce

$$\mathcal{P} = \sigma(\Pi) \subseteq \mathcal{A} \subseteq \mathcal{P} \Rightarrow \mathcal{A} = \mathcal{P}.$$

Now suppose that $H \geq 0$ so previsible. Set

$$H^n = \left(2^{-n} \lfloor 2^n H \rfloor\right) \wedge n$$

$$= \sum_{k=0}^{n2^n - 1} 2^{-n} \cdot k \cdot \mathbf{1}\left(H \in \left[\tfrac{\Sigma - nk}{2^n}, \tfrac{\Sigma - n(k+1)}{2^n}\right)\right) + n \cdot \underbrace{\mathbf{1}(H \geq n)}_{\in \mathcal{P}}.$$

This implies that $H^n$ is a finite linear combination of functions of the form $\mathbf{1}_C$, where $C \in \mathcal{P}$ which in turn implies that $(H^n \cdot A)_t$ is $\mathcal{F}_t$-measurable for all $t$. By the monotone convergence theorem, $(H^n \cdot A)_t \to (H \cdot A)_t$ as $n \to \infty$. For general $H$, write $H = H^+ - H^-$, where $H^\pm = \max(\pm H, 0)$, and use that

$$(H \cdot A)_t = (H^+ \cdot A)_t - (H^- \cdot A)_t \qquad \text{(both } \mathcal{F}_t\text{-measurable).}$$

Step 3. To show that $H \cdot A$ has finite variation, observe that

$$H \cdot A = (H^+ - H^-) \cdot (A^+ - A^-) = (H^+ \cdot A^+ + H^- \cdot A^-) - (H^- \cdot A^+ + H^+ \cdot A^-)$$

is a difference of non-decreasing functions. □

*Next, we will introduce and generalise our theory of stochastic integration to integrating against Martingales.*

Lecture 4

## 3 Local Martingales.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space.

**Definition 3.1.** *We say that* $(\mathcal{F}_t)_{t\geq 0}$ *satisfies the* ***usual conditions*** *if:*

1. $\mathcal{F}_t$ *contains all* $\mathbb{P}$*-null sets.*
2. $(\mathcal{F}_t)_{t\geq 0}$ *is right-continuous:* $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Throughout, assume that $(\mathcal{F}_t)$ satisfies the usual conditions. Recall that an integrable adapted process $X$ is an $(\mathcal{F}_t)$ martingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \text{a.s.}$$

supermartingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s \quad \text{a.s.}$$

submartingale if

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s \quad \text{a.s.}$$

for all $0 \leq s < t$.

A random variable $T$ is called a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. If $X$ is càdlàg and adapted to $(\mathcal{F}_t)$ and we set

$$\mathcal{F}_T = \{E \in \mathcal{F} : E \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

then $X_T$ is an $\mathcal{F}_T$-measurable random variable.

If $X$ is a martingale then $X_t^T = X_{t\wedge T}$ is also a martingale.

**Theorem 3.2** (Optional Stopping Theorem (OST))**.** *Let* $X$ *be an adapted, càdlàg and integrable process. Then the following are equivalent:*

1. $X$ *is a martingale.*
2. $X^T := (X_{t\wedge T})_{t\geq 0}$ *is a martingale for every stopping time* $T$.
3. *For all bounded stopping times* $S \leq T$*, we have*

$$\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S \quad \textit{a.s.}$$

4. *For all bounded stopping times* $T$*, we have that*

$$\mathbb{E}[X_T] = \mathbb{E}[X_0].$$

**Definition 3.3.** *A càdlàg adapted process* $X_t$ *is called a* <u>*local martingale*</u> *if there exists a sequence* $(T_n)_{n\geq 0}$ *of stopping times with* $T_n \nearrow \infty$ *a.s. (non-decreasing), and for every* $n$*, such that the stopped process* $X^{T_n}$ *is a (true) martingale for all* $n \geq 1$*. In this case, we say that* $(T_n)$ ***reduces*** $X$*.*

Note that a MG is a local martingale as any deterministic sequence $T_n \nearrow \infty$ will reduce it.

**Example.** Let $B$ be a standard Brownian motion in $\mathbb{R}^3$. Let $M_t = \frac{1}{|B_t|}$. ( )

(i) $(M_t)_{t\geq 1}$ is $L^2$-bounded: $\sup_{t\geq 1}\mathbb{E}[M_t^2]<\infty$.

(ii) $\mathbb{E}[M_t]\to 0$ as $t\to\infty$.

(iii) $M$ is a supermartingale.

$M$ cannot be a martingale, otherwise its expectation would vanish by (ii), but this cannot be true since $M_t>0$ a.s.

For each $n\geq 1$, set:

$$\begin{aligned} T_n &= \inf\left\{t\geq 1 : |B_t|<\tfrac{1}{n}\right\} \\ &= \inf\left\{t\geq 1 : M_t>n\right\}. \end{aligned}$$

We want to show

1) $(M_{t\wedge T_n})_{t\geq 1}$ is a martingale for all $n$.

2) $T_n\to\infty$ as $n\to\infty$ a.s.

Note that

$$n\leq M_1\Rightarrow T_n=1, \qquad n>M_1\Rightarrow T_n>1.$$

Since $|B_t|$ cannot hit $1/n$ before hitting $|B_1|$, have that $T_n$ is non-decreasing. Now, recall from **Advanced Probability:** $f\in C_0^\infty(\mathbb{R})$

$$f(B_t)-\frac{1}{2}\int_0^t \Delta f(B_s)\,ds \text{ is a martingale.}$$

Note that $f(x)=\frac{1}{|x|}$ is a harmonic function in $\mathbb{R}^3\setminus\{0\}$. Let $(f_n)_{n\geq 1}$ be a sequence of $C_c^\infty(\mathbb{R}^3)$ with $f_n(x)=f(x)$ on $\{|x|\geq\frac{1}{n}\}$. If

$$0<|B_t|<\frac{1}{n},\text{ then } T_n=1 \text{ and so } M_{t\wedge T_n}=M_t \text{ is a martingale.}$$

Since $B_1\neq 0$ a.s., we have that $|B_1|>\frac{1}{n}$ for all $n$ sufficiently large enough, in which case

$$f(B_{t\wedge T_n})=f^n(B_{t\wedge T_n})\quad \forall t\geq 1.$$

Thus:

$$\begin{aligned} M_{t\wedge T_n} &= f(B_{t\wedge T_n})-f(B_t)+f(B_1) \\ &= \left[f(B_{t\wedge T_n})-f(B_t)-\frac{1}{2}\int_1^{t\wedge T_n}\Delta f(B_s)\,\mathrm{d}s\right]+f(B_1) \\ &= \left[f^n(B_{t\wedge T_n})-f^n(B_t)-\frac{1}{2}\int_1^{t\wedge T_n}\Delta f^n(B_s)\,\mathrm{d}s\right]+f^n(B_1) \end{aligned}$$

and so we conslude $M_{T_n}=(M_{t\wedge T_n})_{t\geq 1}$ is a martingale.

We also need to show that $T_n\nearrow\infty$ as $n\to\infty$. Now, as $T_n\leq T_{n+1}$, it remains to show that $\lim_{n\to\infty}T_n=\infty$ a.s.. For each $R$, let

$$S_R=\inf\{t\geq 1 : |B_t|\geq R\}=\inf\{t\geq 1 : M_t<1/R\}.$$

Then $S_R \to \infty$ as $R \to \infty$.

$$\mathbb{P}\left(\lim_{n\to\infty} T_n < \infty\right) \le \mathbb{P}\left(\exists R : T_n < S_R \,\forall n\right) = \lim_{R\to\infty}\lim_{n\to\infty} \mathbb{P}(T_n < S_R).$$

The OST implies

$$\mathbb{E}[M_{T_n \wedge S_R}] = \mathbb{E}[M_1] = N \in (0,\infty).$$

and so the LHS becomes

$$n\mathbb{P}(T_n < S_R) + \frac{1}{R}\mathbb{P}(S_R \le T_n) = \frac{N}{R} \Rightarrow \mathbb{P}(T_n < S_R) = \frac{N - \frac{1}{R}}{n - \frac{1}{R}} \to 0 \text{ as } n \to \infty.$$

Therefore, $(M_t)_{t\ge 0}$, non-negative local martingale but not a martingale, a supermartingale and in $L^2$-bounded.

Observe from the preceding discussion that by only requiring non-negativity, the first two properties actually give that $M$ is a super martingale.

**Proposition 3.4.** *If $X$ is a local martingale, $X_t \ge 0$ for all $t \ge 0$, then $X$ is a supermartingale.*

*Proof.* Suppose that $(T_n)$ is a reducing sequence. Then for any $s \le t$, we know that

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}\left[\lim_{n\to\infty} X_{t\wedge T_n} \mid \mathcal{F}_s\right] \overset{\textbf{(Fatou)}}{\le} \liminf_{n\to\infty} \mathbb{E}[X_{t\wedge T_n} \mid \mathcal{F}_s] = \liminf_{n\to\infty} X_{s\wedge T_n} = X_s \quad \text{a.s.}$$

□

Often work with local martingales instead of martingales, so as to not have to worry about integrability.

Lecture 5

We will now answer the following

1. When is a local MG a MG?
2. Continuous local MGs with finite variation in time.

**Definition 3.5.** *A collection $\mathcal{G}$ of random variables in $L^1(\Omega,\mathcal{F},\mathbb{P})$ is called uniformly integrable (UI) if*

$$\sup_{X\in\mathcal{G}} \mathbb{E}[|X|\mathbf{1}_{|X|>M}] \to 0 \quad \text{as } M \to \infty.$$

**Examples of UI families:**

1. Uniformly bounded random variables: If $\mathcal{G} \subseteq L^1$ is bounded in $L^2$, then $\mathcal{G}$ is UI.
2. $L^p$ bounded for $p > 1$: $\sup_{X\in\mathcal{G}} \mathbb{E}[|X|^p] < \infty$.
3. there exists $Y$ integrable so that $|X| \le Y$ for all $X \in \mathcal{G}$.

**Lemma 3.6.** *Suppose that* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. *Then*

$$\mathcal{X} := \{\mathbb{E}[X \mid \mathcal{G}] : \mathcal{G} \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\}$$

*is also a uniformly integrable family.*

*Proof.* [✍] . □

**Proposition 3.7.** *The following are equivalent:*

*i)* $X$ *is a martingale.*

*ii)* $X$ *is a local martingale and for all* $t > 0$, *the family*

$$\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$$

*is uniformly integrable.*

*Proof.* *i)* ⇒ *ii)*: Suppose $X$ is a martingale. By OST, if $T$ is a stopping time with $T \leq t$, then

$$\mathbb{E}[X_t \mid \mathcal{F}_T] = X_T \Rightarrow X_t \text{ is UI.}$$

*ii)* ⇒ *i)*: Suppose that $X$ is a local martingale and $X_t$ is UI for all $t \geq 0$. To show that $X$ is a martingale, by OST it suffices to show that for all bounded stopping times $T$, we have

$$\mathbb{E}[X_T] = \mathbb{E}[X_0].$$

Let $(T_n)$ be a reducing sequence for $X$ and let $T \leq t$ be a stopping time. Then

$$\mathbb{E}[X_0] = \mathbb{E}[X_0^{T_n}] \overset{\textbf{OST}}{=} \mathbb{E}[X_T^{T_n}] \overset{\textbf{(def'n of } X^{T_n}\textbf{)}}{=} \mathbb{E}[X_{T\wedge T_n}].$$

Since $\{X_{T\wedge T_n} : n \geq 0\}$ is UI and $X_{T\wedge T_n} \to X_T$ a.s.,

$$\text{Advanced Probability } \Rightarrow X_{T\wedge T_n} \to X_T \text{ in } L^1 \text{ as } n \to \infty.$$

Therefore, $\mathbb{E}[X_{T\wedge T_n}] \to \mathbb{E}[X_T]$ as $n \to \infty$. Hence $\mathbb{E}[X_0] = \mathbb{E}[X_T]$. OST finally implies $X$ is a martingale. □

**Corollary 3.1.** *A bounded local martingale is a martingale. More generally, if* $X$ *is a local martingale and there exists* $Y$ *integrable such that* $|X_t| \leq Y$ *for all* $t \geq 0$, *then* $X$ *is a martingale.*

**Theorem 3.8.** *Let* $X$ *be a continuous local martingale with* $X_0 = 0$. *If* $X$ *has finite variation, then* $X \equiv 0$ *a.s.*

*Proof.* Let $V$ be the total variation process for $X$. Then $V_0 = 0$, and $V$ is continuous, adapted and non-decreasing. Let

$$T_n := \inf\{t \geq 0 : V_t = n\}$$

for all $n \in \mathbb{N}$. Then $T_n \nearrow \infty$ as $n \to \infty$, since $X$ has finite variation. Moreover,

$$|X_t^{T_n}| = |X_{t\wedge T_n}| \leq V_{t\wedge T_n} \leq n.$$

Therefore $X^{T_n}$ is a bounded local martingale and hence is a proper MG.
To prove that $X \equiv 0$, note: $X^{T_n} \equiv 0$ for all $T_n \nearrow \infty$ as $n \to \infty$. Fix $n \in \mathbb{N}$, let $Y := X^{T_n}$. $Y$ is a continuous bounded martingale with $Y_0 = 0$. To prove that $Y \equiv 0$, it suffices to show that $\mathbb{E}[Y_t^2] = 0$ for all $t \geq 0$. This implies that $Y_t = 0$ for all $t \geq 0$, $t \in \mathbb{Q}$ a.s., so $Y \equiv 0$ by continuity.
Fix $t \geq 0$, $N \in \mathbb{N}$, let

$$t_k := \frac{k}{N} t \quad \text{for } k \leq N.$$

Compute

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\right] \overset{\text{(MG orthogonality)}}{=} \mathbb{E}\left[\sum_{k=0}^{N-1} (Y_{t_{k+1}} - Y_{t_k})^2\right]$$

$$\leq \mathbb{E}\left[\overbrace{\max_{0\leq k\leq N-1} |Y_{t_{k+1}} - Y_{t_k}|}^{\leq V_{t\wedge T_n} \leq n} \underbrace{\sum_{k=0}^{N-1} |Y_{t_{k+1}} - Y_{t_k}|}_{\leq V_{t\wedge T_n}}\right] \leq n^2.$$

Since $Y$ is continuous,

$$\lim_{N\to\infty} \left(\max_{0\leq k\leq N-1} |Y_{t_{k+1}} - Y_{t_k}|\right) = 0 \quad \text{a.s.}$$

Bounded convergence finally gives $\mathbb{E}[Y_t^2] = 0$. $\square$

**Remark.** *(i) The proof requires continuity, in particular* not *true without continuity.*

*(ii) Theorem implies Brownian motion has infinite variation, so cannot use Lebesgue–Stieltjes integral to define the integral against a BM.*

For continuous local martingales, there is always an explicit way of choosing the reducing sequence.

**Proposition 3.9.** *Let $X$ be a continuous local martingale with $X_0 = 0$. Then*

$$T_n := \inf\{t \geq 0 : |X_t| = n\}$$

*reduces $X$.*

*Proof.* **Step 1:** $T_n$ is a stopping time.
Let $t \geq 0$, then:

$$\{T_n \leq t\} = \{\sup_{0\leq s\leq t} \{|X_s| \geq n\} = \bigcup_{k=1]}^{\infty} \bigcup_{s\in\mathbb{Q}, s\leq t} \overbrace{\{|X_q| \geq n - 1/k\}}^{\in \mathcal{F}_t}.$$

**Step 2:** $T_n \nearrow \infty$ as $n \to \infty$.
Since

$$\sup_{0\leq s\leq t} |X_s(\omega)| < \infty \Rightarrow \text{there exists } n(\omega,t) \in \mathbb{N} \text{ such that } n(\omega,t) \geq \sup_{0\leq s\leq t} |X_s(\omega)|.$$

$$\Rightarrow n \geq n(\omega,t) \Rightarrow T_n(\omega) > t \Rightarrow T_n(\omega) \to \infty \text{ as } n \to \infty.$$

**Step 3:** $(T_n)$ reduces $X$.
Let $(T_n^*)$ be a reducing sequence (exists since $X$ is a local martingale). Then $X^{T_n^*}$ is a martingale for all $n$. Need to show: $X^{T_n}$ is a martingale. The Optional stopping theorem implies $X^{T_n \wedge T_m^*}$ is a martingale for all $n$,

$$X^{T_n} \text{ is a local martingale with reducing sequence } (T_m^*).$$

Since $X^{T_n}$ is in addition bounded, it is a martingale; concluding the proof. □

We now move on to construct the stochastic integral proper.

Lecture 6

# 4 The Stochastic Integral

**Goal:** Be able to integrate against a continuous local MG. How does one construct an integral (Riemann / Lebesgue)?

An integral is a linear map

$$\mathcal{I} : X \to Y \quad \text{where } X, Y \text{ are normed vector spaces.}$$

**Steps:**

① Define it on a dense set $\mathcal{D} \subseteq X$

② Show that it is a continuous linear map:

$$\exists C > 0 \text{ such that } \|\mathcal{I}(f)\|_Y \leq C\|f\|_X \quad \forall f \in \mathcal{D}.$$

$$\Rightarrow \mathcal{I} \text{ extends by continuity to } X.$$

Need to

$$\underbrace{①\ \text{specify } \mathcal{D}, X, Y}_{\text{simple processes, quadratic variation}}, \quad \underbrace{\text{prove } ②}_{\text{Itô isometry}}.$$

**Theorem 4.1.** *Let $X$ be a càdlàg, $L^2$-bounded MG (i.e., $\sup_t \mathbb{E}[X_t^2] < \infty$). Then there exists $X_\infty \in L^2$ such that:*

$$X_t \to X_\infty \quad \textit{a.s. and in } L^2, \quad \textit{and} \quad X_t = \mathbb{E}[X_\infty \mid \mathcal{F}_t] \quad (X_\infty \textit{ is called the final value of } X).$$

**Proposition 4.2** (Doob's $L^2$ inequality)**.** *Let $X$ be a càdlàg, $L^2$-bounded MG. Then:*

$$\mathbb{E}\left[\sup_t |X_t|^2\right] \leq 4\,\mathbb{E}\left[X_\infty^2\right].$$

**Define:**

- $\mathcal{M}^2 = \{L^2\text{-bounded càdlàg MGs}\}$.
- $\mathcal{M}^2_{\mathrm{c}} = \{L^2\text{-bounded, continuous MGs}\}$.

- $\mathcal{M}^2_{c,loc} = \{L^2\text{-bounded, continuous local MGs}\}$.

**Definition 4.3.** *A process $H : [0,\infty) \times \Omega \to \mathbb{R}$ is called a simple process if there exist $0 = t_0 < t_1 < \cdots < t_n$, and bounded, $\mathcal{F}_{t_i}$-measurable random variables $Z_i$, such that:*

$$H_t = \sum_{i=0}^{n-1} Z_i \mathbf{1}_{(t_i,t_{i+1}]}(t).$$

Let $\mathcal{S}$ be the set of simple processes. We will proceed to

- define $\left(\int_0^t H_s\, dM_s\right)$ for $H \in \mathcal{S}$, $M \in \mathcal{M}^2$.
- Extend the integral to more general integrands ($M \in \mathcal{M}^2_{\mathfrak{C}}$).

**Proposition 4.4.** *If $H \in \mathcal{S}$, $M \in \mathcal{M}^2$, then $H \cdot M \in \mathcal{M}^2$. Moreover,*

$$\mathbb{E}[(H \cdot M)^2_\infty] = \sum_{k=0}^{n-1} \mathbb{E}\left[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2\right] \le 4\, \|H\|^2_\infty\, \mathbb{E}\left[(M_\infty - M_0)^2\right].$$

*Proof.* **Step 1:** $H \cdot M$ is a martingale.
Suppose that $t_k \le s < t < t_{k+1}$. Then we have that

$$(H \cdot M)_t - (H \cdot M)_s = Z_k(M_t - M_s),$$

so that

$$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s | \mathcal{F}_s] = Z_k \mathbb{E}[M_t - M_s | \mathcal{F}_s] = 0$$

since $Z_k \in \mathcal{F}_s$ and $M \in \mathcal{M}^2$.
Suppose that $0 \le t_i \le s \le t_j \le t \le t_k$. Then

$$\mathbb{E}[(H \cdot M)_t - (H \cdot M)_s | \mathcal{F}_s]$$

$$= \mathbb{E}\left[\sum_{i=0}^{k-1} Z_i (M_{t_{i+1}} - M_{t_i}) + Z_k(M_t - M_{t_k}) - \left(\sum_{i=0}^{j-1} Z_i(M_{t_{i+1}} - M_{t_i}) + Z_j(M_s - M_{t_j})\right) \middle| \mathcal{F}_s\right].$$

$$= \sum_{i=j+1}^{k-1} \mathbb{E}[Z_i(M_{t_{i+1}} - M_{t_i}) | \mathcal{F}_s] + \mathbb{E}[Z_j(M_{t_j} - M_s)|\mathcal{F}_s] + \mathbb{E}[Z_k(M_t - M_{t_k})|\mathcal{F}_s].$$

Since

$$\begin{aligned} \mathbb{E}[Z_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] &= Z_i \mathbb{E}[M_{t_{i+1}} - M_{t_i}|\mathcal{F}_s] = 0, \quad j+1 \le i \le k-1, \\ \mathbb{E}[Z_j(M_{t_j} - M_s)|\mathcal{F}_s] &= Z_j \mathbb{E}[M_{t_j} - M_s|\mathcal{F}_s] = 0, \\ \mathbb{E}[Z_k(M_t - M_{t_k})|\mathcal{F}_s] &= \mathbb{E}[Z_k \mathbb{E}[M_t - M_{t_k}|\mathcal{F}_{t_k}]|\mathcal{F}_s] = 0\,. \end{aligned}$$

**Step 2:** $H \cdot M$ is $L^2$-bounded.
If $j < k$, then we have that

$$\mathbb{E}\left[Z_j(M_{t_{j+1}} - M_{t_j}) Z_k (M_{t_{k+1}} - M_{t_k})\right] = \mathbb{E}\left[\mathbb{E}[Z_j(M_{t_{j+1}} - M_{t_j})|\mathcal{F}_{t_k}] Z_k(M_{t_{k+1}} - M_{t_k})\right] = 0.$$

So,

$$\mathbb{E}[(H\cdot M)_t^2] = \mathbb{E}\left[\left(\sum_{k=0}^{n-1} Z_k(M_{t_{k+1}} - M_{t_k})\right)^2\right] \overset{\text{MG orthogonality}}{=} \mathbb{E}\left[\sum_{k=0}^{n-1} Z_k^2 (M_{t_{k+1}} - M_{t_k})^2\right]$$

$$\leq \|H\|_\infty^2 \sum_{k=0}^{n-1} \mathbb{E}\left[(M_{t_{k+1}} - M_{t_k})^2\right] \overset{\text{Doob's } L^2 \text{ inequality}}{\leq} 4\,\|H\|_\infty^2\,\mathbb{E}[(M_\infty - M_0)^2].$$

This bound is uniform in $t$, so $H \cdot M$ is $L^2$ bounded, so $H \cdot M \in \mathcal{M}^2$.
**Step 3:**

$$\mathbb{E}[(H\cdot M)_\infty^2] \leq \liminf_{t\to\infty} \mathbb{E}[(H\cdot M)_t^2] \leq \sup_{t\geq 0} \mathbb{E}[(H\cdot M)_t^2] \leq 4\,\|H\|_\infty^2\,\mathbb{E}[(M_\infty - M_0)^2].$$

□

## 4.1 Space of integrators

For $X$ càdlàg and adapted, define the norm:

$$|||X||| = \|X^*\|_{L^2}, \quad X^* = \sup_{t\geq 0} |X_t|.$$

$$\mathcal{C}^2 = \{X \text{ càdlàg, adapted processes } X \text{ with } |||X||| < \infty\}.$$

Define the norm on $\mathcal{M}^2$ is given by

$$|||X||| = \|X_\infty\|_{L^2}.$$

Clearly $\|\cdot\|$ is a seminorm. To see that it is a norm, suppose that

$$|||X||| = \|X_\infty\|_{L^2} = 0 \Rightarrow X_\infty = 0 \text{ a.s.} \Rightarrow X_t = \mathbb{E}[X_\infty|\mathcal{F}_t] = 0 \text{ a.s. for all } t \geq 0.$$

Càdlàg property implies $X \equiv 0$ a.s.
**Setup:**

$$\mathcal{M} = \{\text{càdlàg martingales}\}$$

$$\mathcal{M}_c = \{\text{continuous martingales}\}$$

$$\mathcal{M}_{\text{c, loc}} = \{\text{cont. loc. martingales}\}$$

Lecture 7

**Proposition 4.5.**

***a)*** $(\mathcal{C}^2, \|\!|X\cdot\|\!|$ *is complete.*

***b)*** $\mathcal{M}^2 = \mathcal{M} \cap \mathcal{C}^2$

***c)*** $(\mathcal{M}^2, \|\cdot\|)$ *is a Hilbert space.*

***d)*** $\mathcal{M}_c^2 := \mathcal{M}_c \cap \mathcal{M}^2$ *is a closed subspace.*

*The map*

$$\mathcal{M}^2 \to L^2(\mathcal{F}_\infty), \qquad X \mapsto X_\infty$$

*is an isometry, where*

$$\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t > 0).$$

**Remark.** *We can identify an element of $L^2$ with its final value, so $(\mathcal{M}^2, \|\cdot\|)$ inherits the Hilbert space structure of $(L^2(\mathcal{F}_\infty), \|\cdot\|_{L^2})$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is a closed linear subspace of $(\mathcal{M}^2, \|\cdot\|)$, it is also a Hilbert space. This is the space of processes against which we will integrate.*

*Proof.* **(a)** Suppose that $(X^n)$ is Cauchy with respect to $\|\cdot\|$. Then there exists a subsequence $(X^{n_k})_{k=1}$ of $(X^n)$ such that

$$\sum_k \|X^{n_k} - X^{n_{k+1}}\| < \infty.$$

Thus,

$$\left\| \sum_k \sup_t |X_t^{n_k} - X_t^{n_{k+1}}| \right\|_{L^2} \le \sum_k \|X^{n_k} - X^{n_{k+1}}\| < \infty$$

$$\Rightarrow \sum_{k\ge 0} \sup_{t\ge 0} |X_t^{n_k} - X_t^{n_{k+1}}| < \infty \text{ a.s.}$$

$$\Rightarrow (X^{n_k})_{t\ge 0} \text{ is uniformly Cauchy on } [0,\infty) \text{ a.s., hence converges to a càdlàg limit } X.$$

**NTS:** $X^n \to X$ with respect to $\|\!|\cdot\|\!|$.

$$\|\!|X - X^n\|\!|^2 = \mathbb{E}\left[\sup_{t\ge 0} |X_t - X_t^n|^2\right] = \mathbb{E}\left[\lim_{k\to\infty} \sup_{t\ge 0} |X_t^n - X_t^{n_k}|^2\right]$$

$$\overset{\textbf{Fatou}}{\le} \liminf_{k\to\infty} \mathbb{E}\left[\sup_{t\ge 0} |X_t^n - X_t^{n_k}|^2\right] \le \left(\liminf_{k\to\infty} \|\!|X^n - X^{n_k}\|\!|\right)^2 \to 0 \quad \text{a.s.}$$

Since $X^m$ is Cauchy.

(b) Suppose that $X \in \mathcal{C}^2 \cap \mathcal{M}$. Then

$$\|\!|X\|\!| < +\infty \Rightarrow \sup_{t\ge 0} \|X_t\|_{L^2} \overset{\textbf{Jensen}}{\le} \left\| \sup_{t\ge 0} |X_t| \right\|_{L^2} < \infty \Rightarrow X \in \mathcal{M}^2$$

Suppose that $X \in \mathcal{M}^2$. By Doob's $L^2$-inequality,

$$\|\!|X\|\!| \le 2\|X_\infty\|_{L^2} \Rightarrow 2\|X\| < \infty \Rightarrow X \in \mathcal{C}^2 \cap \mathcal{M}$$

and so

$$\mathcal{M}^2 = \mathcal{M} \cap \mathcal{C}^2$$

(c) Note that $\langle X, Y\rangle := \mathbb{E}[X_\infty Y_\infty]$ defines an inner product on $L^2$. For $X \in \mathcal{M}^2$,

$$|||X||| \leq \|X_\infty\|_{L^2} \leq 2|||X||| \quad \text{(Doob's } L^2\text{-inequality)}$$

which shows that

$$\|\cdot\|\,, |||\cdot||| \text{ are equivalent norms on } \mathcal{M}^2$$

To show that $(\mathcal{M}^2, |||\cdot|||)$ is complete, it suffices to show that $(\mathcal{M}^2, |||\cdot|||)$ is complete. To see this, let $X^n$ be a sequence in $\mathcal{M}^2$ such that

$$|||X^n - X||| \to 0 \text{ as } n \to \infty \text{ where } X \in \mathcal{C}^2$$

(Suffices to show $\mathcal{M}$ is closed.) We know that $X$ is càdlàg, adapted, $L^2$-bounded since $X \in \mathcal{C}^2$.
**NTS:** $X \in \mathcal{M}^2$.
Fix $s \leq t$, we have that

$$\begin{aligned}\|\mathbb{E}[X_t \mid \mathcal{F}_s] - X_s\|_{L^2} &\overset{X^n \text{ is MG}}{=} \|\mathbb{E}\left[X_t - X_t^n \mid \mathcal{F}_s\right] + X_s^n - X_s\|_{L^2} \\ &\leq \|\mathbb{E}\left[X_t - X_t^n \mid \mathcal{F}_s\right]\|_{L^2} + \|X_s^n - X_s\|_{L^2} \\ &\overset{\text{Jensen}}{\leq} \|X_t^n - X_t\|_{L^2} + \|X_s^n - X_s\|_{L^2} \leq 2 \cdot |||X^n - X||| \xrightarrow{n\to\infty} 0\end{aligned}$$

which implies

$$X \in \mathcal{M}^2 \Rightarrow \mathcal{M}^2 \text{ is closed in } \mathcal{C}^2\,.$$

(d) True by definition. □

## 4.2 Space of integrals

Let $(X^n)$ be a sequence of processes. We say that

$$X^n \xrightarrow{\text{ucp}} X \quad \text{uniformly on compact sets in probability (ucp)}$$

if for every $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{s\leq t} |X_s^n - X_s| > \varepsilon\right) \to 0 \quad \text{a.s. as } n \to \infty.$$

**Theorem 4.6.** *Suppose that $M \in \mathcal{M}_{loc}$. There exists a unique (up to indistinguishability), continuous, adapted, non-decreasing process $[M_t]$ such that:*

$$[M]_0 = 0, \quad M^2 - [M] \in \mathcal{M}_{loc}.$$

*Moreover, if we set:*

$$[M]_t^n = \sum_{k=0}^{\lceil 2^n t \rceil - 1} \left( M_{(k+1)2^{-n}} - M_{k2^{-n}} \right)^2,$$

*then*

$$[M]^n \xrightarrow{ucp} [M] \quad \text{as } n \to \infty.$$

*The process $[M]$ is called the quadratic variation of $M$.*

**Examples 4.7.** *Let $B$ be a standard Brownian motion. Then $(B_t^2 - t)_{t \geq 0}$ is a martingale, which implies that $[B]_t = t$. We will prove later that Brownian motion is characterized by this property, i.e., if $M \in \mathcal{M}_{c,loc}$, and $[M]_t = t$ for all $t \geq 0$, then $M$ is a Brownian motion. (Lévy characterization of Brownian motion.)*

*Proof.* Replace $M_t$ with $M_t - M_0$, so without loss of generality $M_0 = 0$.
**Step 1:** *Uniqueness.* Suppose that $A, A'$ are two non-decreasing, continuous, adapted processes satisfying the conditions in the theorem. Then

$$A_t - A'_t = (M_t^2 - A_t) - (M_t^2 - A'_t).$$

LHS: continuous, bounded variation. RHS: process in $\mathcal{M}_{\text{c,loc}} \Rightarrow A - A'$ constant. Since $A_0 = A'_0 = 0 \Rightarrow A = A'$. □

Before we proceed with the proof of existence, we start with a lemma.

Lecture 9

**Lemma 4.8.** *Suppose that $M \in \mathcal{M}_{c,loc}$ is bounded. Then for any $N \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_N < \infty$, we have that:*

$$\mathbb{E}\left[ \left( \sum_{k=0}^{N-1} \underbrace{(M_{t_{k+1}} - M_{t_k})}_{:=\Delta_k} \right)^2 \right] \leq 48 \cdot |||M|||_{L^\infty}^4 .$$

*Proof.* First write

$$\mathbb{E}\left[ \left( \sum_{k=0}^{N-1} \Delta_k \right)^2 \right] \overset{\circledast}{=} \sum_{k=0}^{N-1} \mathbb{E}\left[ (\Delta_k)^4 \right] + 2 \sum_{k=0}^{N-1} \mathbb{E}\left[ \Delta_k^2 \sum_{j=k+1}^{N-1} \Delta_j^2 \right].$$

For each fixed $k$, we have that:

$$\mathbb{E}\left[ \Delta_k^2 \sum_{j=k+1}^{N-1} \Delta_j^2 \right] = \mathbb{E}\left[ \Delta_k^2 \mathbb{E}\left[ \sum_{j=k+1}^{N-1} \Delta_j^2 \middle| \mathcal{F}_{t_{k+1}} \right] \right]$$

$$\overset{\text{MG orthogonality}}{=} \mathbb{E}\left[\Delta_k^2 \mathbb{E}\left[\sum_{j=k+1}^{N-1} \Delta_j^2 \,\middle|\, \mathcal{F}_{t_{k+1}}\right]\right]$$

$$= \mathbb{E}\left[\Delta_k^2 \mathbb{E}\left[(M_{t_N} - M_{t_{k+1}})^2 \,\middle|\, \mathcal{F}_{t_{k+1}}\right]\right] = \mathbb{E}\left[\Delta_k^2 \cdot (M_{t_N} - M_{t_{k+1}})^2\right].$$

Hence,

$$\circledast \le \mathbb{E}\left[\left(\max_{0\le j\le N-1} |M_{t_{j+1}} - M_{t_j}|^2\right) + 2 \cdot \max_{0\le j\le N-1} |M_{t_N} - M_{t_j}|^2 \cdot \left(\sum_{k=0}^{N-1} \Delta_k^2\right)\right]$$

and using the inequality $(a+b)^2 \le 2(a^2+b^2)$, we obtain:

$$\circledast \le 12 \cdot \|\|M\|\|_{L^\infty}^2 \cdot \mathbb{E}\left[\sum_{k=0}^{N-1} \Delta_k^2\right] = 12 \cdot \|\|M\|\|_{L^\infty}^2 \cdot \mathbb{E}\left[\left(\sum_{k=0}^{N-1} \Delta_k\right)^2\right]$$

$$= 12 \cdot \|\|M\|\|_{L^\infty}^2 \cdot \mathbb{E}\left[(M_{t_N} - M_{t_0})^2\right] \le 48 \cdot \|\|M\|\|_{L^\infty}^4.$$

□

*Proof of Theorem 4.6 (Cont'd). Uniqueness*
**WLOG** $M_0 = 0$ (by replacing $M_t$ with $M_t - M_0$ if necessary).
**Step 2:** $M \in \mathcal{M}_c$ bounded ($M \in \mathcal{M}_c^2$). Fix $T > 0$ and set:

$$H_t^n = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}(t).$$

Then $H^n \in \mathcal{S}$ for all $n$, and set

$$X_t^n = (H^n \cdot M)_t = \sum_{k=0}^{\lceil 2^n T\rceil - 1} M_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}}).$$

Then $X^n \in \mathcal{M}_c$, bounded implies $X^n \in \mathcal{M}_c^2$. We will show that $(X^n)$ is Cauchy in $(\mathcal{M}_c^2, \|\cdot\|)$, hence has a limit in $\mathcal{M}_c^2$. Fix $n > m \ge 1$ and write

$$H := H^n - H^m \quad \text{so that} \quad X^n - X^m = (H^n - H^m) \cdot M = H \cdot M.$$

Then,

$$\begin{aligned}
\|X^n - X^m\|^2 &= \mathbb{E}[(H\cdot M)_\infty^2] \\
&= \mathbb{E}[(H\cdot M)_T^2] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\lceil 2^n T\rceil - 1} H_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\lceil 2^n T\rceil - 1} H_{k2^{-n}}^2 (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right] \quad \text{(MG orthogonality)} \\
&\le \mathbb{E}\left[\sup_{t\in[0,T]} |H_t|^2 \cdot \sum_{k=0}^{\lceil 2^n T\rceil - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right]
\end{aligned}$$

$$\leq \left(\mathbb{E}\left[\sup_{t\in[0,T]} |H_t|^4\right]\right)^{1/2} \cdot \left(\mathbb{E}\left[\left(\sum_{k=0}^{\lceil 2^n T\rceil -1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2\right)^2\right]\right)^{1/2}.$$

**First term:** Ⓐ $\sup_{t\in[0,T]} |H_t|^4 = \sup_{t\in[0,T]} |H_t^n - H_t^m|^4 \leq 16 \cdot \|M\|_{L^\infty}^4$.
Ⓑ $\sup_{t\in[0,T]} |H_t^n - H_t^m| \to 0$ as $n, m \to \infty$.
Since $M$ is continuous, by the Bounded Convergence Theorem, first term $\to 0$ as $n, m \to \infty$.

**Second term:**$\leq (48 \cdot \|M\|_{L^\infty}^4)^{1/2} < \infty \Rightarrow \|X^n - X^m\| \to 0 \quad$ as $n, m \to \infty$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete, there exists $Y \in \mathcal{M}_c^2$ such that

$$X_n \to Y \quad \text{as } n \to \infty \text{ in } \mathcal{M}_c^2.$$

For any $n$ and $1 \leq k \leq \lceil 2^n T\rceil$, we have that

$$\begin{aligned} M_{k2^{-n}}^2 - 2X_{k2^{-n}}^n &= \sum_{j=0}^{k-1} (M_{(j+1)2^{-n}} - M_{j2^{-n}})^2 \\ &= [M^n]_{k2^{-n}}. \end{aligned}$$

Hence, for all $n$, $M^2 - 2X^n$ is non-decreasing when restricted to times of the form $\{k2^{-n} : 1 \leq k \leq \lceil 2^n T\rceil\}$. To prove the same is also true for $M^2 - 2Y$, it suffices to show that $X^n \to Y$ a.s. uniformly, at least along a subsequence. This follows from the equivalence of norms $\|\cdot\|$, $|||\cdot|||$. Set $[M]_t := M_t^2 - 2Y_t$. Then $[M]$ is continuous, adapted, non-decreasing and

$$M^2 - [M] = 2Y \in \mathcal{M}_c.$$

Can extend to all times by applying the above $T = k$, $\forall k \in \mathbb{N}$. *Uniqueness*implies the process obtained with $T = k$, $T = k+1$ restricted to $[0, k]$ is the same.

<u>**Step 3:**</u> $\quad [M^n] \to [M]$ ucp as $n \to \infty$.
Observe that

$$X^n \to Y \quad \text{in } (\mathcal{M}_c^2, \|\cdot\|) \Rightarrow \sup_{0\leq t\leq T} |X_t^n - Y_t| \to 0 \text{ as } n \to \infty \text{ in } L^2$$

since $\|\cdot\|$, $|||\cdot|||$ are equivalent which implies $\sup_{0\leq t\leq T} |X_t^n - Y_t| \to 0$ in probability.
Now, $[M]_t^n = M_{2^{-n}\lceil 2^n t\rceil}^2 - 2X_{2^{-n}\lceil 2^n t\rceil}^n$. So,

$$\sup_{0\leq t\leq T} |[M]_t^n - [M]_t| \leq \sup_{0\leq t\leq T} \left|M_{2^{-n}\lceil 2^n t\rceil}^2 - M_t^2\right| \tag{4.1}$$

$$+ 2 \cdot \sup_{0\leq t\leq T} \left|X_{2^{-n}\lceil 2^n t\rceil^n} - Y_{2^{-n}\lceil 2^n t\rceil}\right| + 2 \cdot \sup_{0\leq t\leq T} \left|Y_{2^{-n}\lceil 2^n t\rceil} - Y_t\right|. \tag{4.2}$$

Each term on RHS converges to zero in probability and so we obtain the ucp convergence.

Lecture 9 <u>**Step 4:**</u> Let $M_n \in \mathcal{M}_{c,loc}$. *"Localization argument"*.
For each $n \in \mathbb{N}$, let $\tau_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then $(\tau_n)$ reduces $M$ and $M_n := M^{\tau_n}$ is a bounded MG for all $n$. Therefore, there exists a unique continuous, adapted and non-decreasing process $[M^{T_n}]$ such that

$$[M^{T_n}]_0 = 0 \quad \text{and} \quad (M^{T_n})^2 - [M^{T_n}] \in \mathcal{M}_{c,loc}.$$

Let $A^n := [M^{T_n}]$. By uniqueness, $(A^{n+1}_{t\wedge T_n}, A^n_t)$ are indistinguishable. Let $A$ be the process such that

$$A_{t\wedge T_n} = A^n_t\,,\ \text{for all } n \geq 1\,.$$

Then $M^2_{t\wedge T_n} - A_{t\wedge T_n} \in \mathcal{M}$ for all $n \in \mathbb{N}$ and so $M^2 - A \in \mathcal{M}_{c,loc}$ with reducing sequence $(T_n)$ giving $[M] = A$.
We know that $[M^{T_k}]^n \to [M^{T_k}]$ in ucp as $n \to \infty$ for all $k$. In other words, for all

$$\varepsilon, T > 0: \quad \mathbb{P}\left[\sup_{0\leq t\leq T} |[M^{T_k}]^n_t - [M^{T_k}]_t| > \varepsilon\right] \to 0 \quad \text{as } n \to \infty.$$

On $\{T_k \leq T\}$, $[M^n]_t = [M^{T_k}]^n_t$ and $[M]_t = [M^{T_k}]_t$. Thus,

$$\mathbb{P}\left[\sup_{0\leq t\leq T} |[M]^n_t - [M]_t| > \varepsilon\right] \leq \mathbb{P}[T_k \leq T] + \mathbb{P}\left[\sup_{0\leq t\leq T} |[M^{T_k}]^n_t - [M^{T_k}]_t| > \varepsilon\right] \to 0$$
$$\text{as } n \to \infty,\ \text{then } k \to \infty.$$

LHS $\to 0$ as $n \to \infty$. ☐

**Theorem 4.9.** *Let $M \in \mathcal{M}^2_c$. Then $M^2 - [M]$ is a UI martingale.*

*Proof.* Let $T_n := \inf\{t \geq 0 : [M]_t \geq n\}$ for $n \in \mathbb{N}$. Then $T_n \nearrow \infty$ as $n \to \infty$, $T_n$ is a stopping time, $[M]_{t\wedge T_n} \leq n$ and (noting $M^{T_n} \in \mathcal{M}_{c,loc}$, for all $n \geq 1$)

$$|M^2_{t\wedge T_n} - [M]_{t\wedge T_n}| \leq n + \sup_{u\geq 0} M^2_u.$$

By Doob's inequality the RHS is integrable and so

$$M^2_{t\wedge T_n} - [M]_{t\wedge T_n} \in \mathcal{M}_c\,.$$

The Optional Stopping Theorem (OST) also gives

$$\mathbb{E}\left[M^2_{t\wedge T_n} - [M]_{t\wedge T_n}\right] = 0 \Rightarrow \mathbb{E}\left[[M]_{t\wedge T_n}\right] = \mathbb{E}\left[M^2_{t\wedge T_n}\right].$$

Send $t \to \infty$; the Monotone Convergence Theorem (MCT) implies

$$\text{LHS} \overset{t\to\infty}{\longrightarrow} \mathbb{E}\left[[M]_{T_n}\right],$$

and the Dominated Convergence Theorem (MCT) also implies

$$\text{RHS} \overset{t\to\infty}{\longrightarrow} \mathbb{E}\left[M^2_{T_n}\right].$$

and so

$$\mathbb{E}\left[[M]_{T_n}\right] = \mathbb{E}\left[M^2_{T_n}\right].$$

Finally, send $n \to \infty$. MCT implies the LHS converges to $\mathbb{E}\left[[M]_\infty\right]$, and the RHS converges to

$$\mathbb{E}\left[M^2_\infty\right] \Rightarrow \mathbb{E}\left[[M]_\infty\right] = \mathbb{E}\left[M^2_\infty\right] < \infty \Rightarrow \mathbb{E}\left[[M]_\infty\right] \text{ is integrable.}$$

Moreover,

$$|M^2_t - [M]_t| \leq \sup_{u\geq 0} M^2_u + [M]_\infty.$$

So we conclude the RHS is integrable $\Rightarrow M^2 - [M] \in \mathcal{M}_c$ and UI as it is dominated by an integrable r.v. ☐

## 4.3 The Space $L^2(M)$, $M \in \mathcal{M}_c^2$

Recall that $\mathcal{P}$ = previsible $\sigma$-algebra:

$$\mathcal{P} = \sigma(\{E \times (s,t] : E \in \mathcal{F}_s,\ s < t\}).$$

For $A \in \mathcal{P}$, define

$$\mu(A) = \mathbb{E}\left[\int_0^\infty \mathbf{1}_A(\omega, s)\, d[M]_s\right].$$

Then $\mu$ is a measure on $(\Omega \times [0,\infty), \mathcal{P})$. Moreover, it is uniquely determined by

$$\mu(E \times (s,t]) = \mathbb{E}\left[\mathbf{1}_E\left([M]_t - [M]_s\right)\right] \quad \text{for } s < t,\ E \in \mathcal{F}_s,$$

since $\mathcal{P}$ is generated by sets of this form and they form a $\pi$-system. If $H \geq 0$ is previsible, then:

$$\int_{\Omega \times [0,\infty)} H\, d\mu = \mathbb{E}\left[\int_0^\infty H_s\, d[M]_s\right].$$

**Definition 4.10.** *Let* $L^2(\mu) := L^2(\Omega \times [0,\infty), \mathcal{P}, \mu)$*.*
*Write* $\|H\|_{L^2(\mu)} = \|H\|_\mu := \left(\mathbb{E}\left[\int_0^\infty H_s^2\, d[M]_s\right]\right)^{1/2}$*. Then* $L^2(\mu)$ *= previsible processes with* $\|H\|_\mu < \infty$*, a Hilbert space. This is the space of integrands.*

**Remark.** $(L^2(\mu), \|\cdot\|_\mu)$ *depends on* $M$*, since* $\mu$ *depends on* $M$*, but the simple processes are always*

$$\mathcal{S} \subseteq L^2(M) \quad \forall M \in \mathcal{M}_c^2.$$

*(here* $\mathcal{S}$ *denotes simple processes)*

## 4.4 Itô integrals

Recall that for

$$H = \sum_{k=0}^{n-1} Z_k \mathbf{1}_{(t_k, t_{k+1}]} \in \mathcal{S}, \quad M \in \mathcal{M}_c^2,$$

we set

$$(H \cdot M)_t := \sum_{k=0}^{n-1} Z_k (M_{t_{k+1} \wedge t} - M_{t_k \wedge t}) \in \mathcal{M}_c^2.$$

This map defines a map

$$L^2(M) \supseteq \mathcal{S} \longrightarrow \mathcal{M}_c^2.$$

We will prove that it defines an isometry between

$$(L^2(\mu), \|\cdot\|_\mu) \quad \text{and} \quad (\mathcal{M}_c^2, \|\cdot\|),$$

when restricted to $\mathcal{S} \subset L^2(M)$. (Itô isometry). Indeed, compute

$$\begin{aligned} \|H \cdot M\|^2 &= \|(H \cdot M)_\infty\|_{L^2}^2 \quad \text{(see calculation from before)} \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left[Z_k^2 (M_{t_{k+1}} - M_{t_k})^2\right]. \end{aligned}$$

Since $M^2 - [M]$ is a martingale, we have that

$$\begin{aligned}\mathbb{E}\left[Z_k^2(M_{t_{k+1}} - M_{t_k})^2\right] &= \mathbb{E}\left[Z_k^2\mathbb{E}\left[(M_{t_{k+1}} - M_{t_k})^2 \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2\mathbb{E}\left[M_{t_{k+1}}^2 - M_{t_k}^2 \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2\mathbb{E}\left[[M]_{t_{k+1}} - [M]_{t_k} \mid \mathcal{F}_{t_k}\right]\right] \\ &= \mathbb{E}\left[Z_k^2([M]_{t_{k+1}} - [M]_{t_k})\right].\end{aligned}$$

Hence,

$$\begin{aligned}\|H \cdot M\|^2 &= \mathbb{E}\left[\sum_{k=0}^{n-1} Z_k^2([M]_{t_{k+1}} - [M]_{t_k})\right] \\ &= \mathbb{E}\left[\int_0^\infty H_s^2 \, d[M]_s\right] = \|H\|_M^2.\end{aligned}$$

Lecture 10

**Theorem 4.11** (Itô Isometry). *There exists a unique isometry $I : L^2(M) \to \mathcal{M}_c^2$ such that*

$$I(H) = H \cdot M \quad \textit{for all simple } H (\in \mathcal{S}).$$

**Definition:** For $M \in \mathcal{L}^2$, $H \in L^2(M)$, let

$$H \cdot M := I(H) \quad \text{where } I \text{ is from the theorem.}$$

To prove the theorem, we first prove that the simple processes are dense in $L^2(M)$.

**Lemma 4.12.** *Let $\nu$ be any finite measure on $\mathcal{P}$. Then $\mathcal{S}$ is dense in $L^2(\mathcal{P}, \nu)$. In particular, if $M \in \mathcal{M}_{c,loc}$ and we take $\nu = \mu$, we have that $\mathcal{S}$ is dense in $L^2(M)$.*

*Proof.* Since $H \in \mathcal{S} \Rightarrow \|H \cdot M\|_{L^\infty} < \infty$, it follows that $\mathcal{S} \subseteq L^2(\mathcal{P}, \nu)$. Let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}$ in $L^2(\mathcal{P}, \nu)$. We wish to show: $\overline{\mathcal{S}} = L^2(\mathcal{P}, \nu)$. Let $\mathcal{A} := \{A \in \mathcal{P} : \mathbf{1}_A \in \overline{\mathcal{S}}\}$.
We wish to show: $\mathcal{A} = \mathcal{P}$. It is obvious that $\mathcal{A} \subseteq \mathcal{P}$. To see why the other direction holds, note that:

(A) contains the $\pi$-system $\{E \times (s, t] : E \in \mathcal{F}_s,\ s < t\}$, which generates $\mathcal{P}$,

(B) $\mathcal{A}$ is a $\lambda$-system.

By Dynkin's lemma, it follows that $\mathcal{P} \subseteq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{P}$. Thus, the lemma follows since linear combinations of such indicators are dense in $L^2(\mathcal{P}, \nu)$. □

*Proof of Itô Isometry.* Take $H \in L^2(M)$. The above lemma implies there exists $(H^n) \subset \mathcal{S}$ such that

$$\|H^n - H\|_{L^2(M)} \to 0 \quad \text{as } n \to \infty.$$

This implies $(H^n)$ is a Cauchy sequence with respect to $\|\cdot\|_{L^2(M)}$.
**Need to show:** $I(H^n)$ is Cauchy with respect to $\|\cdot\|$.

$$\begin{aligned}\|I(H^n) - I(H^m)\| &= \|H^n \cdot M - H^m \cdot M\| \quad \text{(linearity)} \\ &= \|(H^n - H^m) \cdot M\| = \|H^n - H^m\|_M \quad \text{(isometry)} \\ &\to 0 \quad \text{as } n, m \to \infty.\end{aligned}$$

Therefore, $(I(H^n))$ converges with respect to $\|\cdot\|$ to an element in $\mathcal{M}_c^2$. Since $(\mathcal{M}_c^2, \|\cdot\|)$ is complete, set $I(H)$ to be this element.

**NTS:** $I$ is well-defined.
Suppose that $(K^n) \subset \mathcal{S}$ converges to $H$ with respect to $\|\cdot\|_{L^2(M)}$. Then

$$\begin{aligned}\|I(H^n) - I(K^n)\| &= \|H^n \cdot M - K^n \cdot M\| \\ &= \|H^n - K^n\|_M \le \|H^n - H\|_M + \|K^n - H\|_M \to 0\end{aligned}$$

as $n \to \infty$, so that the limits of $I(H^n), I(K^n)$ are indistinguishable.
**NTS:** $I$ is an isometry $L^2(M) \to \mathcal{M}_c^2$
$(H^n) \subset \mathcal{S}, H^n \to H \in L^2(M), \|I(H)\| = \lim \|H^n \cdot M\| = \lim \|H^n\|_M = \|H\|_M$. □

From now on, we write

$$I(H)_t = (H \cdot M)_t = \int_0^t H_s \,\mathrm{d}M_s$$

This process $H \cdot M$ is the Itô (stochastic) integral of $H$ with respect to $M$.

**Extensions:** Our goal now is to extend the definition of $H \cdot M$ to the setting that $H$ is locally bounded and $M \in \mathcal{M}_{c,\text{loc}}$. Need to understand how the integral behaves under stopping.

**Proposition 4.13.** *Let $H \in \mathcal{S}, M \in \mathcal{M}$. Then for any stopping time $T$, we have that*

$$(H \cdot M^T) = (H \cdot M)^T.$$

*Proof.* We have that:

$$\begin{aligned}(H \cdot M^T)_t &= \sum_{k=0}^{n-1} Z_k (M^T_{t\wedge t_{k+1}} - M^T_{t\wedge t_k}) \\ &= \sum_{k=0}^{n-1} Z_k \left(M_{t\wedge(t_{k+1}\wedge T)} - M_{t\wedge(t_k\wedge T)}\right) \\ &= (H \cdot M)_{t\wedge T} = (H \cdot M)^T_t.\end{aligned}$$

□

**Proposition 4.14.** *Let $M \in \mathcal{M}_c^2$, $H \in L^2(M)$, and $T$ a stopping time. Then*

$$(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M = (H \cdot M^T).$$

*Proof.* First note that if $H \in L^2(M)$, then $H \cdot \mathbf{1}_{(0,T]} \in L^2(M)$ and $H \in L^2(M^T)$, so the integrals make sense.

**Step 1:** Let $H \in \mathcal{S}, M \in \mathcal{M}_c^2$, and $T$ takes on finitely many values. Then $H \cdot \mathbf{1}_{(0,T]} \in \mathcal{S}$ and

$$(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M = H \cdot M^T.$$

**Step 2:** Let $H \in \mathcal{S}, M \in \mathcal{M}_c^2$, and $T$ a general stopping time. Previous proposition implies $(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M$. **Need to show:** $(H \cdot M)^T = (H \cdot \mathbf{1}_{(0,T]}) \cdot M$. Will prove via an approximation argument.
For $m, n \in \mathbb{N}$, let $T_{n,m} = (2^{-n}\lceil 2^n T\rceil) \wedge m$. Then $T_{n,m}$ takes finitely many values and $T_{n,m} \searrow T \wedge m$ as $n \to \infty$. Thus,

$$\left\| H \cdot \mathbf{1}_{(0,T_{n,m}]} - H \cdot \mathbf{1}_{(0,T\wedge m]} \right\|_{L^2(M)}^2 = \mathbb{E}\left[\int_0^\infty H_t^2 \cdot \mathbf{1}_{(T_{n,m}, T\wedge m]} \, d[M]_t\right] \to 0,$$

as $n \to \infty$ by the Dominated Convergence Theorem, with dominating function $H_t^2$. Therefore, $(H \cdot \mathbf{1}_{(0,T_{n,m}]}) \cdot M \to (H \cdot \mathbf{1}_{(0,T\wedge m]}) \cdot M$ in $\mathcal{M}_c^2$ as $n \to \infty$.
**Step 3:**

$$\text{LHS} = (H \cdot M)^{T_{n,m}}, \quad (H \cdot M)^{T_{n,m}} \to (H \cdot M)^{T \wedge m}$$

pointwise almost surely by continuity of $H \cdot M$. Thus,

$$(H \cdot \mathbf{1}_{(0,T\wedge m)}) \cdot M \to (H \cdot M)^{T\wedge m}.$$

Repeat the same argument, send $n \to \infty$

$$\Rightarrow H \cdot \mathbf{1}_{(0,T]} \cdot M = (H \cdot M)^T.$$

**Step 3:** Let $H \in L^2(M), M \in \mathcal{M}_c^2, T$ a general stopping time.
Let $(H^n)$ be a sequence in $\mathcal{S}$ with $H^n \to H$ in $L^2(M)$. Then,

$$\begin{aligned}
\left\| (H^n \cdot M)^T - (H \cdot M)^T \right\|_{\mathcal{M}_c^2} &= \| (H^n \cdot M)_T - (H \cdot M)_T \|_{L^2} \\
&\le \left\| \sup_{t \le T} (H^n \cdot M)_t - (H \cdot M)_t \right\|_{L^2} \\
&\le 2 \cdot \| (H^n \cdot M)_\infty - (H \cdot M)_\infty \|_{L^2} \quad \text{(Doob's } L^2 \text{ inequality)} \\
&= 2 \cdot \| (H^n - H) \cdot M \| = 2 \cdot \| H^n - H \|_M \to 0 \text{ as } n \to \infty
\end{aligned}$$

(by Itô isometry) and so

$$(H^n \cdot M)^T \to (H \cdot M)^T \text{ in } \mathcal{M}_c^2.$$

On the other hand,

$$\begin{aligned}
\left\| H^n \cdot \mathbf{1}_{(0,T]} - H \cdot \mathbf{1}_{(0,T]} \right\|_M^2 &= \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2 \cdot \mathbf{1}_{(0,T]} \, d[M]_t\right] \\
&\le \mathbb{E}\left[\int_0^\infty (H_t^n - H_t)^2 \, d[M]_t\right] = \| H^n - H \|_M^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence,

$$H^n \cdot \mathbf{1}_{(0,T]} \cdot M \to H \cdot \mathbf{1}_{(0,T]} \cdot M \text{ in } \mathcal{M}_c^2 \text{ by the Itô isometry.}$$

Since $H^n \cdot \mathbf{1}_{(0,T]} \cdot M = (H^n \cdot M)^T$ for all $n$, we have that

$$(H \cdot M)^T = H \cdot \mathbf{1}_{(0,T]} \cdot M.$$

□

**NTS:** $(H \cdot M)^T = (H \circ M^T)$. Assume there exists $(H^n)$ in $\mathcal{S}$ such that $H^n \to H$ in $L^2(\mu)$.

$$\begin{aligned}\|H^n - H\|_{\mu^T}^2 &= \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2 \, d[M^T]_s\right] \\ &= \mathbb{E}\left[\int_0^\infty (H_s^n - H_s)^2 \cdot \mathbf{1}_{(0,T]} \, d[M]_s\right] \\ &\leq \|H^n - H\|_\mu^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

$$\Rightarrow H^n \circ M^T \to H \circ M^T \text{ in } \mathcal{M}_c^2 \text{ by Itô isometry.}$$

Since $(H^n \cdot M)^T = H^n \circ M^T$ for all $n$, we get that

$$(H \cdot M)^T = (H \circ M^T). \qquad \square$$

**Definition 4.15.** *We say that a previsible process $H$ is locally bounded if there exists a sequence $(S_n)_{n \in \mathbb{N}}$ of stopping times where $S_n \nearrow \infty$ as $n \to \infty$ and $H \cdot \mathbf{1}_{(0,S_n]}$ is bounded for all $n$.*

**Remark.** *Every continuous adapted process is previsible and locally bounded.*

**Definition 4.16.** *Let $H$ be a locally bounded, previsible process with $H \cdot \mathbf{1}_{[0,S_n]}$ bounded for all $n$, where $(S_n)$ is a sequence of stopping times with $S_n \nearrow \infty$ as $n \to \infty$. Let $M \in \mathcal{M}_{c,loc}$ with $M_0 = 0$ and let*

$$S_n' := \inf\{t \geq 0 : |M_t| \geq n\}$$

*so that $M^{S_n'} \in \mathcal{M}_c^2$ for all $n$. Let $T_n := S_n \wedge S_n'$, and set*

$$(H \cdot M)_t := (H\mathbf{1}_{(0,T_n]} \cdot M^{T_n})_t, \quad \forall t \in [0, T_n].$$

*Using the previous proposition, this definition is well-defined, and is consistent with the Itô integral with $M \in \mathcal{M}_c^2$, $H \in L^2(M)$.*

**Proposition 4.17.** *Let $M \in \mathcal{M}_{c,loc}$, $H$ locally bounded and previsible, then $H \cdot M \in \mathcal{M}_{c,loc}$ where the sequence $(T_n)$ is a reducing sequence. Moreover, for any stopping time $T$, we have that*

$$(H \cdot M)^T = H\mathbf{1}_{(0,T]} \cdot M = H \cdot M^T.$$

*Proof.* That $H \cdot M \in \mathcal{M}_{c,\text{loc}}$ with reducing sequence $(T_n)$ follows from the definition of $H \cdot M$. For any stopping time $T$,

$$(H \cdot M)^T = \lim_{n \to \infty} (H\mathbf{1}_{(0,T_n]} \cdot M^{T_n})^T \quad \text{(pointwise limit)}.$$

By the previous proposition,

$$(H \cdot M)^T = \lim_{n \to \infty} (H\mathbf{1}_{(0,T]} \cdot \mathbf{1}_{(0,T_n]} \cdot M^T = H \cdot \mathbf{1}_{(0,T]} \circ M.$$

The same argument shows that $(H \cdot M)^T = H \cdot M^T$. $\square$

Lecture 12 Today we will show

$$[H \cdot M] = H^2 \cdot [M], \quad H \cdot (K \cdot M) = (HK) \cdot M,$$

for semimartingales.

**Proposition 4.18.** *Let $M \in \mathcal{M}_{c,loc}$ and $H$ locally bounded and previsible. Then*

$$\underbrace{[H \cdot M]}_{\text{Itô}} = \underbrace{H^2 \cdot [M]}_{\text{Lebesgue-Stieltjes}} .$$

*Proof.* Suppose that $T$ is a bounded stopping time. Then $H, M$ are uniformly bounded. Then

$$\begin{aligned} \mathbb{E}\left[(H \cdot M)_T^2\right] &= \mathbb{E}\left[\left((H \cdot \mathbf{1}_{(0,T]}) \cdot M\right)_\infty^2\right] \\ &= \mathbb{E}\left[\left(H^2 \cdot \mathbf{1}_{(0,T]} \cdot [M]\right)_\infty\right] && \text{(Itô isometry)} \\ &= \mathbb{E}\left[\left(H^2 \cdot [M]\right)_T\right]. \end{aligned}$$

**OST:** $(H \cdot M)^2 - H^2 \cdot [M] \in \mathcal{M}_c$. Uniqueness of quadratic variation implies

$$[H \cdot M] = H^2 \cdot [M].$$

Now assume that $H$ is locally bounded, previsible, and $M \in \mathcal{M}_{c,\text{loc}}$. Let $(T_n)$ be a sequence of stopping times so that $H \cdot \mathbf{1}_{(0,T_n]}, M^{T_n}$ are bounded, and $T_n \to \infty$ as $n \to \infty$. Then

$$\begin{aligned} [H \cdot M] &= \lim_{n\to\infty} [H \cdot M]^{T_n} \\ &= \lim_{n\to\infty} [(H \cdot M)^{T_n}] && \text{(uniqueness of quadratic variation)} \\ &= \lim_{n\to\infty} [(H\mathbf{1}_{(0,T_n]}) \cdot M] \\ &= \lim_{n\to\infty} H^2 \mathbf{1}_{(0,T_n]} \cdot [M^{T_n}] \\ &= H^2 \cdot [M] \quad \text{(applying MCT).} \quad \square \end{aligned}$$

□

Since $H \cdot M \in \mathcal{M}_{c,\text{loc}}$ for $M \in \mathcal{M}_{c,\text{loc}}$, $H$ locally bounded, previsible, we can integrate against it.

**Proposition 4.19.** *Let $M \in \mathcal{M}_{c,loc}, H, K$ locally bounded, previsible. Then:*

$$H \cdot (K \cdot M) = (HK) \cdot M.$$

*Proof.* Elementary to check that this holds for $H, K$ simple processes, . Note that by linearity in each argument, it suffices to check for $H, K$ consisting of single time intervals and noticing that for $0 \le s'' < s' < t'$, $0 < s < t$,

$$\mathbf{1}_{(s''\wedge t', t'\wedge t]} - \mathbf{1}_{(s\wedge t', t'\wedge s'')} = \mathbf{1}_{(s''\wedge t', t')} \cdot \mathbf{1}_{(s', t]}$$

Now suppose that $H, K, M$ are uniformly bounded. **NTS:** $H \in L^2(K \cdot M)$, $HK \in L^2(M)$.

$$\|H\|_{L^2(K\cdot M)}^2 = \mathbb{E}\left[(H^2 \cdot [K \cdot M])_\infty\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left(H^2 \cdot (K^2 \cdot [M])\right)_\infty\right] \\
&= \mathbb{E}\left[\left((HK)^2 \cdot [M]\right)_\infty\right] && \text{(Lebesgue–Stieltjes)} \\
&= \|HK\|^2_{L^2(M)} \\
&\leq \min\left\{\|H\|^2_\infty \|K\|^2_{L^2(M)}, \|K\|^2_\infty \|H\|^2_{L^2(M)}\right\} < \infty.
\end{aligned}
$$

Let $(H^n), (K^n)$ be sequences in $\mathcal{S}$ which converge to $H, K$ in $L^2(M)$ and where $(H^n), (K^n)$ uniformly bounded. Then

$$H^n \cdot (K^n \cdot M) = (H^n K^n) \cdot M.$$

Then

$$
\begin{aligned}
\|H^n \cdot (K^n \cdot M) - H \cdot (K \cdot M)\| &\leq \|(H^n - H) \cdot (K^n \cdot M)\| + \|H \cdot ((K^n - K) \cdot M)\| \\
&= \|H^n - H\|_{L^2(K^n \cdot M)} + \|H\|_{L^2((K^n - K)\cdot M)} && \text{(Itô iso} \\
&= \|(H^n - H) \cdot K^n\|_{L^2(M)} + \|H \cdot (K^n - K)\|_{L^2(M)} \\
&\leq \|K^n\|_\infty \|H^n - H\|_{L^2(M)} + \|H\|_\infty \|K^n - K\|_{L^2(M)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

A similar argument shows $(H^n K^n) \cdot M \to (HK) \cdot M$ as $n \to \infty$ in $\mathcal{M}_c$ yielding

$$H \cdot (K \cdot M) = (HK) \cdot M \quad \text{(bounded case)}.$$

Now suppose that $H, K$ are locally bounded, previsible and $M \in \mathcal{M}_{c,\text{loc}}$. Let $(T_n)$ be a sequence of stopping times so that

$$H\mathbf{1}_{[0,T_n]},\ K\mathbf{1}_{[0,T_n]},\ M^{T_n} \text{ are bounded and } T_n \nearrow \infty \text{ as } n \to \infty.$$

Then

$$HK\mathbf{1}_{[0,T_n]} \cdot M^{T_n} = \left(H\mathbf{1}_{[0,T_n]}\right) \cdot \left(K\mathbf{1}_{[0,T_n]} \cdot M^{T_n}\right).$$

Also,

$$K\mathbf{1}_{[0,T_n]} \cdot M^{T_n} = (K \cdot M)^{T_n}.$$

Hence,

$$H\mathbf{1}_{[0,T_n]} \cdot (K\mathbf{1}_{[0,T_n]} \cdot M)^{T_n} = H\mathbf{1}_{[0,T_n]} \cdot (K \cdot M)^{T_n} = (H \cdot (K \cdot M))^{T_n} \to H \cdot (K \cdot M) \quad \text{as } n \to \infty.$$

Also,

$$(HK\mathbf{1}_{[0,T_n]}) \cdot M^{T_n} = (HK \cdot M)^{T_n} \to (HK \cdot M) \quad \text{as } n \to \infty$$

which finally gives

$$H \cdot (K \cdot M) = (HK) \cdot M.$$

□

**Remark.** *We have repeatedly used a "localisation" argument to reduce everything to the setting of a bounded integrand and martingale. This is a standard procedure; will omit in later arguments.*

# 5 Semimartingales

**Definition 5.1.** *A continuous, adapted process $X$ is a* semimartingale *if it can be decomposed as*

$$X = X_0 + M + A$$

*where $M \in \mathcal{M}_{c,loc}$, $A$ is of finite variation, and $M_0 = A_0 = 0$.*

**"Doob–Meyer decomposition":** For a continuous semi-martingale $X = X_0 + M + A$, define the quadratic variation by $[X]_t := [M]_t$. Justified since once can compute ( )

$$\sum_{k=0}^{\lceil 2^n t\rceil - 1} \left( X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}} \right)^2 \xrightarrow[n\to\infty]{ucp} [M]_t \, .$$

**Definition 5.2.** *For $H$ locally bounded and previsible, and $X = X_0 + M + A$ a continuous semimartingale, define (Here, the first term is the Itô integral, the second is Lebesgue–Stieltjes.)*

$$H \cdot X := H \cdot M + \int H_s \, dA_s \, .$$

*Then $H \cdot X$ is also a semimartingale.*

**Proposition 5.3.** *Let $X$ be a continuous semimartingale and $H$ locally bounded, left-continuous and adapted. Then:*

$$\sum_{k=0}^{\lceil 2^n t\rceil - 1} H_{k\cdot 2^{-n}} (X_{(k+1)\cdot 2^{-n}} - X_{k\cdot 2^{-n}}) \xrightarrow[n\to\infty]{ucp} (H \cdot X)_t$$

*Proof.* . Hint: use a localisation argument first. Show that the Itô integral of $H$ can be approximated by discretely approximating $H$ by simple processes. □

Lecture 13

**Summary of the Stochastic Integral**

**Step 1:** $H \in \mathcal{S}$, $H_t = \sum_{k=0}^{n-1} Z_k \cdot \mathbf{1}_{(t_k, t_{k+1}]}(t)$,
$Z_k$ bounded, $\mathcal{F}_{t_k}$-measurable, $M \in \mathcal{M}_c^2$ set:

$$(H \cdot M)_t = \sum_{k=0}^{n-1} Z_k (M_{t\wedge t_{k+1}} - M_{t\wedge t_k}).$$

Then $H \cdot M \in \mathcal{M}_c^2$.

**Step 2:** Equip $\mathcal{M}_c^2$ with a Hilbert space structure with norm $\|M\| = \|M_\infty\|_{L^2}$, $M \in \mathcal{M}_c^2$.

**Step 3:** Establish the existence of $[M] \in \mathcal{M}_{\text{c, loc}}$, where $[M]$ is the unique adapted, non-decreasing continuous process with $[M]_0 = 0$ so that $M^2 - [M] \in \mathcal{M}_{\text{c, loc}}$.

**Step 4:** For $M \in \mathcal{M}_c^2$, use $[M]$ to define a Hilbert space $(L^2(M), \|\cdot\|_M$ where

$$\|H\|_M = \left( \mathbb{E}\left[ \int_0^\infty H_s^2 \, d[M]_s \right] \right)^{1/2}$$

**Step 5:** Extend the integral to $H \in L^2(M)$, $M \in \mathcal{M}_c^2$ using the Itô isometry:

$$\|H \cdot M\| = \|H\|_{\mathcal{H}_M}$$

$H \cdot M \in \mathcal{M}_c^2$ for all $H \in L^2(M)$, $M \in \mathcal{M}_c^2$.

**Step 6:** Extended to $H$ locally bounded & previsible, $M \in \mathcal{M}_{\text{c, loc}}$ by setting

$$(H \cdot M)_t = (H \mathbf{1}_{[0,\tau_n]} \cdot M^{\tau_n})_t \quad \forall t \leq \tau_n$$

**Step 7:** Extend to $H$ locally bounded, previsible and $X = X_0 + M + A$ a continuous semimartingale by setting

$$H \cdot X = \underbrace{H \cdot M}_{\text{Itô}} + \underbrace{H \cdot A}_{\text{Lebesgue–Stieltjes}}$$

then $H \cdot X$ is a continuous semimartingale.

## Stochastic Calculus

**Definition 5.4.** *For $M, N \in \mathcal{M}_{c,loc}$, define the* covariation *of $M, N$ by setting:*

$$[M, N] := \frac{1}{4}\left([M+N] - [M-N]\right).$$

*(Polarization identity). Note that: $[M, M] = [M]$.*

**Theorem 5.5.** *Let $M, N \in \mathcal{M}_{c,loc}$. Then:*

(a) *$[M, N]$ is the unique process (up to indistinguishability), continuous, adapted, finite-variation process with $[M, N]_0 = 0$, so that $MN - [M, N] \in \mathcal{M}_{c,loc}$.*

(b) *For $n \in \mathbb{N}$, set*

$$[M, N]_t^n := \sum_{k=0}^{\lceil 2^n t \rceil - 1} \left(M_{(k+1)2^{-n}} - M_{k2^{-n}}\right)\left(N_{(k+1)2^{-n}} - N_{k2^{-n}}\right).$$

*Then $[M, N]_t^n \to [M, N]_t$ as $n \to \infty$, almost surely and locally uniformly in $t$.*

(c) *If $M, N \in \mathcal{M}_c^2$, then $MN - [M, N]$ is a UI martingale.*

(d) *For $H$ locally bounded, previsible,*

$$[H \cdot M, N] + [M, H \cdot N] = 2H \cdot [M, N].$$

*Proof.* (a) $MN = \frac{1}{4}\left((M+N)^2 - (M-N)^2\right)$. So

$$\circledast\, MN - [M, N] = \frac{1}{4}\Big((M+N)^2 - [M+N] - (M-N)^2 + [M-N]\Big), \quad \in \mathcal{M}_{c,\text{loc}}.$$

Therefore, $MN - [M, N] \in \mathcal{M}_{c,\text{loc}}$. By definition, $[M, N]$ is continuous, adapted and finite-variation (difference of non-decreasing functions). Same argument used to prove the uniqueness of covariation.

(b) Note that

$$\begin{array}{cccc} [M,N]_t^n = & \frac{1}{4} ( [M+N]_t^n & - & [M-N]_t^n ) \\ \downarrow\text{ucp} & \downarrow\text{ucp} & & \downarrow\text{ucp} \\ [M,N] & [M+N] & & [M-N] \end{array}$$

So $[M,N]_t^n \to [M,N]_t$ ucp.

(c) $MN - [M,N]$ is a UI martingale for $M, N \in \mathcal{M}_c^2$, follows from the identity $\circledast$ and the corresponding property for quadratic variation.

(d)
$$[H \cdot (M+N)] = H^2 \cdot [M+N],$$
so
$$[H \cdot M, H \cdot N] = H \cdot [M,N].$$
Moreover,
$$(H+1)^2 \cdot [M,N] = [(H+1) \cdot M, (H+1) \cdot N]$$
by bilinearity ( ✍ )
$$= [H \cdot M + M, H \cdot N + N]$$
$$= [H \cdot M, H \cdot N] + [H \cdot M, N] + [M, H \cdot N] + [M,N],$$
and
$$\begin{aligned}(H+1)^2 \cdot [M,N] &= (H^2 + 2H + 1) \cdot [M,N] \\ &= H^2 \cdot [M,N] + 2H \cdot [M,N] + [M,N].\end{aligned}$$
giving
$$2H \cdot [M,N] = [M, H \cdot N] + [H \cdot M, N].$$ □

**Proposition 5.6** (Kunita–Watanabe identity)**.** *Let $M, N \in \mathcal{M}_{c,\text{loc}}$, $H$ locally bounded, previsible. Then*
$$[H \cdot M, N] = H \cdot [M,N].$$

*Proof.* **NTS:** $[H \cdot M, N] = [N, H \cdot M]$, as then we can apply part (d) of the previous theorem. Now, use that
$$\begin{aligned}(H \cdot M)N - [H \cdot M, N] &\in \mathcal{M}_{c,\text{loc}}, \\ M(H \cdot N) - [M, H \cdot N] &\in \mathcal{M}_{c,\text{loc}}.\end{aligned}$$
We will show that
$$(H \cdot M)N - M(H \cdot N) \in \mathcal{M}_{c,\text{loc}}.$$
This suffices, since then $[H \cdot M, N] - [M, H \cdot N] \in \mathcal{M}_{c,\text{loc}}$ with finite variation and starts from 0, so
$$[H \cdot M, N] = [M, H \cdot N].$$
**Localisation:** WLOG $M, N \in \mathcal{M}_c^2$, $H$ bounded.
By optional stopping, it suffices to show that for bounded stopping time $T$,
$$\mathbb{E}[(H \cdot M)_T N_T] = \mathbb{E}[M_T (H \cdot N)_T].$$

LHS $= \mathbb{E}[(H \cdot M)^T_\infty N^T_\infty]$, RHS $= \mathbb{E}[M^T_\infty (H \cdot N)^T_\infty]$. So it suffices to show that

$$\mathbb{E}[(H \cdot M)_\infty N_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$$

for all $M, N \in \mathcal{M}^2_c$, bounded $H$. Suppose now that $H = Z\mathbf{1}_{(s,t]}$, $Z$ $\mathcal{F}_s$-measurable, bounded. We then compute

$$\begin{aligned}\mathbb{E}[(H \cdot M)_\infty N_\infty] &= \mathbb{E}\left[Z(M_t - M_s)N_\infty\right] \\ &= \mathbb{E}[ZM_t\mathbb{E}[N_\infty \mid \mathcal{F}_t] - ZM_s\mathbb{E}[N_\infty \mid \mathcal{F}_s]] \\ &= \mathbb{E}\left[Z(M_tN_t - M_sN_s\right] \\ &= \mathbb{E}[M_\infty(H \cdot N)_\infty],\end{aligned}$$

Same argument the same argument gives

$$\mathbb{E}[M_\infty (H \cdot N)_\infty] = \mathbb{E}[M_\infty (H \cdot N)_\infty]$$

for $H = \sum Z\mathbf{1}_{(s,t]}$. Linearity gives ⊛ for $H \in \mathcal{S}$.

Lecture 14

Suppose now that $H$ is a bounded predictable process. Then there exists a sequence $(H^n) \subset \mathcal{S}$ so that $H^n \to H$ in $L^2(M), L^2(N)$ (in the lemma where we showed that $\mathcal{S}$ are dense in $L^2(\mathbb{P}, \nu)$, $\nu$ finite, to be given by $\nu(E) = \mathbb{E}\left[\int_0^\infty \mathbf{1}_E(\mathrm{d}[M]_s + \mathrm{d}[N]_s)\right])$. Hence,

$$H^n \cdot M \to H \cdot M, \quad H^n \cdot N \to H \cdot N \text{ in } \|\cdot\|\text{-norm}$$

and so

$$H^n \cdot M)_\infty \to (H \cdot M)_\infty \text{ and in } L^2$$

and

$$(H^n \cdot N)_\infty \to (H \cdot N)_\infty \quad \text{as } n \to \infty$$

Thus,

$$\begin{aligned}\|\mathbb{E}[(H^n \cdot M)_\infty N_\infty] - \mathbb{E}[(H \cdot M)_\infty N_\infty]\|_{L^1} &\overset{\text{C-S}}{\leq} \|(H^n \cdot M)_\infty - (H \cdot M)_\infty\|_{L^2} \|N_\infty\|_{L^2} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus,

$$\mathbb{E}[(H^n \cdot M)_\infty N_\infty] \overset{n\to\infty}{\longrightarrow} \mathbb{E}[(H \cdot M)_\infty N_\infty]$$

Same works with $M, N$ swapped which finally gives ⊛. □

**Definition 5.7.** *For continuous semi-martingales $X, Y$, define $[X, Y]$ to be the covariation of their martingale parts.*

- *This is justified as*

$$[X, Y]_t^n = \sum_{k=0}^{\lceil 2^n t \rceil - 1} (X_{(k+1)2^{-n}} - X_{k2^{-n}})(Y_{(k+1)2^{-n}} - Y_{k2^{-n}})$$

$$\xrightarrow{ucp} [X, Y]_t \text{ as } n \to \infty$$

- *Kunita–Watanabe also holds for semi-martingales.*

**Proposition 5.8.** *Let $X, Y$ be independent semi-martingales. Then their covariation $[X, Y] = 0$.*

*Proof.* [✍] . ☐

## 5.1 Itô's formula

**Theorem 5.9** (Integration by parts)**.** *Let $X, Y$ be continuous semi-martingales. Then*

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + [X, Y]_t. \qquad \circledast$$

*Proof.* Note that the integrals are well-defined since any continuous adapted process is locally bounded and predictable.
Note that for $s \leq t$, we have

$$\begin{aligned} X_t Y_t - X_s Y_s &= X_s(Y_t - Y_s) + Y_s(X_t - X_s) \\ &\quad + (X_t - X_s)(Y_t - Y_s). \end{aligned}$$

Since the LHS and RHS of identity $\circledast$ are continuous, it suffices to prove the result for $t$ of the form

$$t = m \cdot 2^{-j}, \quad m, j \in \mathbb{N}, \quad (n \geq j),$$

$$\begin{aligned} X_t Y_t - X_0 Y_0 = \sum_{k=0}^{m \cdot 2^{n-j} - 1} & (X_{k \cdot 2^{-n}}(Y_{(k+1)\cdot 2^{-n}} - Y_{k \cdot 2^{-n}}) + Y_{k \cdot 2^{-n}}(X_{(k+1)\cdot 2^{-n}} - X_{k \cdot 2^{-n}}) \\ & + (X_{(k+1)\cdot 2^{-n}} - X_{k \cdot 2^{-n}})(Y_{(k+1)\cdot 2^{-n}} - Y_{k \cdot 2^{-n}})). \\ & \xrightarrow{\text{ucp}} (X \cdot Y)_t + (Y \cdot X)_t + [X, Y]_t \text{ as } j \to \infty. \end{aligned}$$

☐

Note that the $[X, Y]$ term does not appear if either $X, Y$ are independent or if $X$ or $Y$ does not have a martingale part.

**Theorem 5.10** (Itô's Formula)**.** *Let* $(X^1, \dots, X^d)$ *where each* $X^i$*, for* $1 \le i \le d$*, is a continuous semi-martingale. Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be* $C^2$*. Then,*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d[X^i, X^j]_s.$$

**Remark.** 1. *Integration by parts is a special case of Itô's formula with* $f(x, y) = x \cdot y$*.*

2. *For* $d = 1$*, Itô's formula reads:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2} \int_0^t f''(X_s)\, d[X]_s.$$

*It is possible to derive this using Taylor expansions, since:*

$$f(X_t) = f(X_0) + \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \Big( f(X_{(k+1)2^{-n}}) - f(X_{k2^{-n}}) \Big)$$

$$= f(X_0) + \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} f'(X_{k2^{-n}})(X_{(k+1)2^{-n}} - X_{k2^{-n}}) + \frac{1}{2} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} f''(X_{k2^{-n}})(X_{(k+1)2^{-n}} - X_{k2^{-n}})^2 + \textit{error}.$$

$$\longrightarrow f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2} \int_0^t f''(X_s)\, d[X]_s \quad (\textit{ucp as } n \to \infty).$$

*We will prove it a different way, since the extra error term is inconvenient to deal with.*

**Examples 5.11.** 1. *Let* $X = B$*, a standard Brownian motion, and* $f(x) = x^2$*. Then:*

$$f(X_t) = f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, d[B]_s$$

$$= 0 + \int_0^t 2B_s\, dB_s + \frac{1}{2} \int_0^t 2\, ds = 2 \int_0^t B_s\, dB_s + t$$

*which gives*

$$B_t^2 - t = 2 \int_0^t B_s\, dB_s \in \mathcal{M}_{c,loc}.$$

2. *Let* $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ *be* $C^{1,2}$*, and define*

$$X_t = (t, B_t^1, \dots, B_t^d)$$

*where* $B_t^1, \dots, B_t^d$ *are independent Brownian motions. By Itô's formula:*

$$f(t, B_t) - f(0, B_0) = \int_0^t \left( \frac{\partial}{\partial s} + \frac{1}{2} \Delta \right) f(s, B_s)\, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, B_s)\, dB_s^i \in \mathcal{M}_{c,loc}.$$

*Here,* $\Delta$ *is the Laplacian in the spatial coordinates.*

If $f$ does not depend on $t$ and is harmonic in spatial variables, then $f(B_t) \in \mathcal{M}_{c,\text{loc}}$. If $f$ is bounded, then $f(B_t)$ is a martingale.

Lecture 15

*Proof (Itô's Formula).* We are doing the proof for $d = 1$; the case $d > 1$ is just notationally more cumbersome but the same argument essentially applies, . Let

$$X = X_0 + M + A$$

and let $V$ be the total variation of $A$. Let

$$T_r = \inf\{t \geq 0 : |X_t| + V_t + [M]_t > r\}$$

for each $r > 0$. Then $(T_r)$ is a sequence of stopping times with $T_r \nearrow \infty$ as $r \to \infty$.
It suffices to prove the formula on $[0, T_r]$ for each $r > 0$. Let $\mathcal{A}$ be the subset of $C_c^2(\mathbb{R})$ such that the formula holds. We show $\mathcal{A} = C_c^2(\mathbb{R})$.
We will prove this by showing

(a) $\mathcal{A}$ contains $f(x) \equiv 1$, $f(x) \equiv x$.

(b) $\mathcal{A}$ is a vector space.

(c) $\mathcal{A}$ is an algebra, i.e., $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$.

(d) If $(f_n) \subset \mathcal{A}$ with

$$f_n \to f \text{ in } C^2(\overline{B_r}) \text{ for each } r > 0$$

(where $B_r = \{x \in \mathbb{R} : |x| < r\}$), then $f \in \mathcal{A}$.

Here, $f_n \to f$ in $C^2(\overline{B_r})$ means that with

$$\Delta_{n,r} := \sup_{x \in \overline{B_r}} |f_n - f| + \sup_{x \in \overline{B_r}} |f_n' - f'| + \sup_{x \in \overline{B_r}} |f_n'' - f''|,$$

we have $\Delta_{n,r} \to 0$ as $n \to \infty$ for each $r > 0$.

(a), (b), (c) imply that polynomials are in $\mathcal{A}$. The Weierstrass approximation theorem gives that polynomials are dense in $C^2(\overline{B_r})$ $\forall r > 0$, so (d) implies that $\mathcal{A} = C_c^2(\mathbb{R})$. That (a), (b) hold is easy to see, .
Proof of (c): Suppose $f, g \in \mathcal{A}$. Let $F_t = f(X_t)$, $G_t = g(X_t)$. Itô's formula holds for $f, g$ give that $F, G$ are continuous semi-martingales. Integration by parts also gives

$$F_t G_t - F_0 G_0 = \int_0^t F_s \, dG_s + \int_0^t G_s \, dF_s + [F, G]_t.$$

Since Itô's formula holds for $f, g$, we have:

$$\int_0^t F_s \, dG_s = \int_0^t F_s d\left(\int_0^s g'(X_u) \, dX_u + \frac{1}{2}\int_0^s g''(X_u) \, d[X]_u\right). \tag{1}$$

$$\overset{\text{K-W}}{=} \int_0^t f(X_s) g'(X_s) \, dX_s + \frac{1}{2}\int_0^t f(X_s) g''(X_s) \, d[X]_s \tag{2}$$

Also,

$$\int_0^t G_s \, dF_s = \int_0^t f'(X_s) g(X_s) \, dX_s + \frac{1}{2}\int_0^t f''(X_s) g(X_s) \, d[X]_s \tag{3}$$

$$[F,G]_t = [f(X),g(X)]_t = [f'(X)\cdot X,\ g'(X)\cdot X] \quad \text{(by def. of cov. and Itô formula)}$$
$$= \int_0^t f'(X_s)g'(X_s)\,d[X]_s \quad \text{(Kunita-Watanabe)} \tag{4}$$

Plug (2)–(4) into (1) gives Itô's formula for $fg$, i.e., $fg \in \mathcal{A}$.
Proof of (d): Suppose that $(f_n)$ is a sequence in $\mathcal{A}$ and $f_n \to f$ in $C^2(\overline{B_r})$ for all $r > 0$.
WTS: Itô's formula for $f$, i.e., $f \in \mathcal{A}$. Since Itô's formula holds for $f_n$:

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)\,dA_s + \frac{1}{2}\int_0^t f_n''(X_s)\,d[M]_s + \int_0^t f_n'(X_s)\,dM_s.$$

Finite variation part:

$$\int_0^{t\wedge T_r} \left(f_n'(X_s) - f'(X_s)\right)\,dV_s + \frac{1}{2}\int_0^{t\wedge T_r} \left(f_n''(X_s) - f''(X_s)\right)\,d[M]_s$$
$$\le \Delta_{n,r}\cdot\left(V_{t\wedge T_r} + \frac{1}{2}[M]_{t\wedge T_r}\right) \le 2r\cdot\Delta_{n,r} \to 0 \quad \text{as } n\to\infty$$

which implies that

$$\overset{n\to\infty}{\Longrightarrow} \int_0^{t\wedge T_r} f_n'(X_s)\,dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f_n''(X_s)\,d[M]_s \to \int_0^{t\wedge T_r} f'(X_s)\,dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f''(X_s)\,d[M]_s \quad \text{uniformly in } t.$$

MG part: $M^r \in \mathcal{M}_c^2$ since $[M]_T \le r$.

$$\left\|\left(f_n'(X)\cdot M\right)^{T_r} - \left(f'(X)\cdot M\right)^{T_r}\right\|_2^2 = \mathbb{E}\left[\int_0^{T_r} \left(f_n'(X_s) - f'(X_s)\right)^2 d[M]_s\right]$$
$$\le \Delta_{n,r}^2\cdot\mathbb{E}\left[[M]_{T_r}\right] \le r\Delta_{n,r}^2 \to 0 \quad \text{as } n\to\infty$$

which implies that

$$\left(f_n'(X)\cdot M\right)^{T_r} \to \left(f'(X)\cdot M\right)^{T_r} \text{ in } \mathcal{M}_c \text{ as } n\to\infty$$

finally giving

$$f(X_{t\wedge T_r}) = f(X_0) + \int_0^{t\wedge T_r} f'(X_s)dA_s + \frac{1}{2}\int_0^{t\wedge T_r} f''(X_s)d[M]_s + \int_0^{t\wedge T_r} f'(X_s)dM_s.$$

□

## 5.2 Stratonovich Integral

Let $X, Y$ be continuous semi-martingales. The Stratonovich integral of $X$ against $Y$ is defined as:

$$\int_0^t X_s\partial Y_s := \underbrace{\int_0^t X_s\,dY_s}_{\text{(Itô)}} + \frac{1}{2}[X,Y]_t.$$

This is essentially a 'midpoint approximation' since one can show

$$\sum_{k=0}^{\lfloor 2^n t\rfloor - 1}\left(\frac{X_{k2^{-n}} + X_{(k+1)2^{-n}}}{2}\right)\left(Y_{(k+1)2^{-n}} - Y_{k2^{-n}}\right) \overset{ucp}{\to} \int_0^t X_s\partial Y_s.$$

**Proposition 5.12.** *Let $X^1, \dots, X^d$ be continuous semi-martingales and let $f : \mathbb{R}^d \to \mathbb{R}$ be $C^3$. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_s) \partial X_s^i$$

In particular, integration by parts is given by:

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s.$$

This shows that the Stratonovich integral satisfies the usual rules of calculus. But the Stratonovich integral against $\mathcal{M}_c \cap \mathcal{M}_{loc}$ is <u>not</u> in $\mathcal{M}_{\text{c, loc}}$.

For example,

$$\int_0^t B_s \partial B_s = \int_0^t B_s \, dB_s + \frac{1}{2} t = \frac{1}{2} B_t^2 \notin \mathcal{M}_{\text{c, loc}}$$

*for $B$ a standard Brownian motion.*

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**Proposition 5.13.** *Let $X^1, \dots, X^d$ be continuous semi-martingales, $X = (X^1, \dots, X^d)$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be $C^3$. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_s) \partial X_s^i$$

In particular,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s \partial Y_s + \int_0^t Y_s \partial X_s$$

*Proof.* $d = 1$: $d > 1$ is similar,  . Itô's formula gives,

$$(1) \quad f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s$$

$$(2) \quad f'(X_t) = f'(X_0) + \int_0^t f''(X_s) dX_s + \frac{1}{2} \int_0^t f^{(3)}(X_s) d[X]_s$$

$$[f'(X), X]_t \overset{(2)}{=} [f''(X) \cdot X, X]_t = f''(X) \cdot [X]_t \quad \text{(Kunita–Watanabe)}$$

giving

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} [f'(X), X]_t = f(X_0) + \int_0^t f'(X_s) \partial X_s \,.$$

□

Before we proceed with some applications of the theory developed so far, we will make the following notational conventions.

<u>Shorthand:</u>

$$Z_t = Z_0 + \int_0^t H_s dX_s \quad \Leftrightarrow \quad \mathrm{d}Z_t = H_t \, \mathrm{d}X_t$$

$$Z_t = Z_0 + \int_0^t H_s \partial X_s \quad \Leftrightarrow \quad \partial Z_t = H_t \partial X_t$$

$$Z_t = [X,Y]_t = \int_0^t d[X,Y]_s \quad \Leftrightarrow \quad \partial Z_t = \mathrm{d}X_t \, \mathrm{d}Y_t$$

Computational rules

$$\begin{aligned}
H_t d(K_t \, \mathrm{d}X_t) &= (H_t K_t) \, \mathrm{d}X_t \quad \text{[Iterated integral]} \\
H_t d(X_t \, \mathrm{d}Y_t) &= d(H_t X_t) \, \mathrm{d}Y_t \quad \text{[Kunita–Watanabe]} \\
d(X_t Y_t) &= X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + d[X,Y]_t \quad \text{[Integration by parts]} \\
df(X_t) &= \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_t) \, \mathrm{d}X_t^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(X_t) \, \mathrm{d}X_t^i \, \mathrm{d}X_t^j \quad \text{[Itô's formula]}
\end{aligned}$$

# 6 Applications

**Theorem 6.1** (Lévy Characterisation)**.** *Let $X^1, \ldots, X^d \in \mathcal{M}_{c,\mathrm{loc}}$, and set $X = (X^1, \ldots, X^d)$. Suppose $X_0 = 0$, and*

$$[X^i, X^j]_t = \delta_{ij} t \quad \forall i, j, \ t \geq 0.$$

*Then $X$ is a standard Brownian motion.*

*Proof.* We need to show: for all $0 \leq s \leq t < \infty$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the law of $\mathcal{N}(0, (t-s)\mathrm{Id})$, where Id is the $d \times d$ identity matrix. quivalently, for all $\theta \in \mathbb{R}^d$,

$$\mathbb{E}\left[\exp(i\langle \theta, X_t - X_s \rangle) \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\theta|^2(t-s)\right)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $|\theta|^2 = \langle \theta, \theta \rangle$. To see this, let $A \in \mathcal{F}_s$, $\mathbb{P}(A) \neq 0$ and define the probability measure

$$\mathbb{P}_A(\cdot) := \mathbb{P}(A)^{-1} \mathbb{P}(\cdot \cap A).$$

Then, by the tower property,

$$\mathbb{E}_{\mathbb{P}_A}\left[\exp(i\langle \theta, X_t - X_s \rangle)\right] = \mathbb{E}\left[\exp(i\langle \theta, X_t - X_s \rangle)\right]$$

which implies that the law of $X_t - X_s$ under $\mathbb{P}_A$ is the same under $\mathbb{P}$. hence, for all bounded and measurable $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\left[\mathbf{1}_A \cdot f(X_t - X_s)\right] = \mathbb{P}(A) \cdot \mathbb{E}\left[f(X_t - X_s)\right]$$

which implies $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$.
For $\theta \in \mathbb{R}^d$, set $Y_t = \langle \theta, X_t \rangle = \sum_{j=1}^d \theta_j X_t^j$. Then $Y \in \mathcal{M}_{c,\mathrm{loc}}$ since $\mathcal{M}_{c,\mathrm{loc}}$ is a vector space. Moreover,

$$[Y]_t = [Y,Y]_t = \left[\sum_{j=1}^d \theta_j X^j, \sum_{k=1}^d \theta_k X^k\right]_t = \sum_{j,k=1}^d \theta_j \theta_k [X^j, X^k]_t = |\theta|^2 t.$$

Let

$$Z_t = \exp\left(iY_t + \frac{1}{2}[Y]_t\right) = \exp\left(i\langle\theta, X_t\rangle + \frac{1}{2}|\theta|^2 t\right).$$

By Itô's formula applied to $W_t = iY_t + \frac{1}{2}[Y]_t$, with $f(w) = e^w \in C^2$, we have:

$$\mathrm{d}Z_t = Z_t\left(i\ \mathrm{d}Y_t + \frac{1}{2}d[Y]_t\right) - \frac{1}{2}Z_t\, d[Y]_t = iZ_t\ \mathrm{d}Y_t.$$

which implies $Z \in \mathcal{M}_{c,\text{loc}}$ since $Y \in \mathcal{M}_{c,\text{loc}}$. Since $Z$ is bounded on $[s,t]$ for $t < \infty$, $Z \in \mathcal{M}$. Thus, $\mathbb{E}[Z_t \mid \mathcal{F}_s] = Z_s$ and so

$$\mathbb{E}\left[\exp(i\langle\theta, X_t - X_s\rangle) \mid \mathcal{F}_s\right] = \exp\left(-\frac{1}{2}|\theta|^2(t-s)\right).$$

□

**Theorem 6.2** (Dubins–Schwarz)**.** *Let $M \in \mathcal{M}_{c,\text{loc}}$ with $M_0 = 0$, $[M]_\infty = \infty$. Set*

$$\tau_s := \inf\{t \geq 0 : [M]_t > s\}, \quad B_s := M_{\tau_s}, \quad \mathcal{G}_s := \mathcal{F}_{\tau_s}.$$

*Then $(\tau_s)$ is an $(\mathcal{F}_t)$-stopping time and $[M]_{\tau_s} = s$ for all $s \geq 0$. Moreover, $B$ is a $(\mathcal{G}_s)$-Brownian motion with $M_t = B_{[M]_t}$.*
This means that every continuous local martingale starting from 0 is a time-change of a standard Brownian motion.

*Proof.* Since $[M]$ is continuous and adapted, $\tau_s$ is a stopping time for each $s \geq 0$. Since $[M]_\infty = \infty$, $\tau_s$ is a finite stopping time $\forall s \geq 0$. Moreover, $(\mathcal{G}_s)$ is a filtration since if $S, T$ are stopping times with $s \leq t$, then $\tau_s \leq \tau_t \Rightarrow \mathcal{F}_{\tau_s} \subseteq \mathcal{F}_{\tau_t} \Rightarrow \mathcal{G}_s \subseteq \mathcal{G}_t$.

Step 1: $B$ is adapted to $(\mathcal{G}_s)$. NTS: $M_{\tau_s}$ is $\mathcal{F}_{\tau_s}$-measurable $\forall s \geq 0$.

Recall that, ( ) if $X$ is càdlàg, adapted, and $T$ a stopping time, then $X_T \mathbf{1}_{\{T<\infty\}}$ is $\mathcal{F}_T$-measurable.
Now, apply for $X = M$ and $T = \tau_s$, and use that $\mathbb{P}(\tau_s < \infty) = 1$.

Step 2: $B$ is continuous.
Since $s \mapsto \tau_s$ is non-decreasing and càdlàg, it follows that $B$ is càdlàg (since $B_s = M_{\tau_s}$). To prove that $B$ is continuous, it suffices to show

$$B_{s^-} = B_s \quad \forall s \geq 0 \quad \iff \quad M_{\tau_s^-} = M_{\tau_s} \quad \forall s \geq 0.$$

where $\tau_s^- := \inf\{t \geq 0 : [M]_t = s\}$. If $\tau_s = \tau_s^-$, there is nothing to prove. If $\tau_s > \tau_s^-$, then $[M]_t$ is constant on $[\tau_s^-, \tau_s]$.
NTS: If $[M]_t$ is constant on any interval, then $M_t$ is constant as well. For each rational $q \in \mathbb{Q}$, define

$$S_q := \inf\{t > q : [M]_t > [M]_q\}.$$

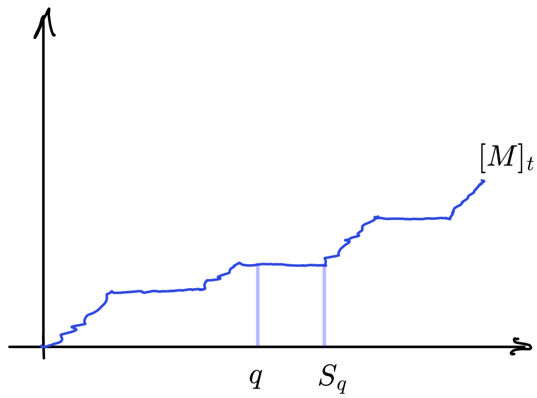


Figure 1: Illustration of times $S_q$.

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We continue working on step 2, which is the continuity of $B$. Need to prove that if $[M]$ is constant on a given interval, then $M$ is constant on the same interval. By localisation, WLOG, $M \in \mathcal{M}_c^2$. Suppose that $q \in \mathbb{Q}, q > 0$. It suffices to show that $M$ is a.s. constant on each $[q, S_q]$. We know that $M^2 - [M]$ is a local martingale since $M \in \mathcal{M}_c$. By OST, we have that:

$$\mathbb{E}\left[M_{S_q}^2 - [M]_{S_q} \mid \mathcal{F}_{S_q}\right] = M_q^2 - [M]_q. \quad \circledast$$

Since $M \in \mathcal{M}_c^2$, we also have that

$$\text{(MG orthog.)} \quad \mathbb{E}\left[(M_{S_q} - M_q)^2 \mid \mathcal{F}_{S_q}\right] = \mathbb{E}[M_{S_q}^2 - M_q^2 \mid \mathcal{F}_{S_q}]$$

$$(*) \quad = \mathbb{E}\left[M_{S_q}^2 - [M]_{S_q} \mid \mathcal{F}_{S_q}\right] = 0 \quad \text{since } [M]_{S_q} = [M]_q.$$

Therefore $M_{S_q} - M_q = 0$ a.s. which implies $M$ is a.s. constant on $[q, S_q]$ since for all $t \geq q$,

$$M_{t \wedge S_q} = \mathbb{E}[M_{S_q} \mid \mathcal{F}_t] = \mathbb{E}[M_q \mid \mathcal{F}_q] = M_q \text{ , a.s.}$$

Step 3: $B$ is a $(\mathcal{G}_s)$-BM.
Fix $s > 0$. Then we know that $[M^{\tau_s}]_\infty = [M]_{\tau_s} = s$. Therefore $M^{\tau_s} \in \mathcal{M}_c^2$, since $\mathbb{E}[[M^{\tau_s}]_\infty] < \infty$. Therefore $(M^2 - [M])^{\tau_s}$ is a UI MG. By OST, for $0 \leq t \leq s < \infty$, we have that:

(i) $\mathbb{E}[B_s \mid \mathcal{G}_t] = \mathbb{E}[M_{\tau_s} \mid \mathcal{F}_{\tau_t}] = M_{\tau_t} = B_t$.

(ii) $\mathbb{E}[B_s^2 - s \mid \mathcal{G}_t] = \mathbb{E}[(M^2 - [M])_{\tau_s} \mid \mathcal{F}_{\tau_t}] = M_{\tau_t}^2 - [M]_{\tau_t} = B_t^2 - t$

Thus, (i) implies $B \in \mathcal{M}_c$, and (ii) implies $[B]_s = s$ and so

$$B \text{ is a } (\mathcal{G}_s)\text{-BM by the Lévy characterisation.}$$

□

Dubins–Schwarz requires $[M]_\infty = \infty$. One can also provide an extension thereof for the case that $[M]_\infty < \infty$:

**Theorem 6.3.** *$M \in \mathcal{M}_{loc}, M_0 = 0$. Let $\beta$ be a BM which is independent of $M$. Set:*

$$B_s = \begin{cases} M_{\tau_s} & \text{if } s \leq [M]_\infty \\ M_\infty + \left(\beta_s - \beta_{[M]_\infty}\right) & \text{if } s > [M]_\infty \end{cases}$$

*Then $B$ is a standard BM and $M_t = B_{[M]_t}$ for all $t \geq 0$.*

**Examples.**

(i) Let $B$ be a standard BM, $h$ deterministic, measurable in $L^2([0,\infty))$. Let

$$M_t = \int_0^t h(s)\, dB_s.$$

Then $M_0 = 0$, $M \in \mathcal{M}_{\text{loc}}$, and

$$[M]_t = \int_0^t h(s)^2\, ds.$$

Moreover,

$$M_\infty \overset{d}{=} B_{\int_0^\infty h(s)^2 ds} \quad \text{(Dubins–Schwarz)} \sim \mathcal{N}(0, \|h\|_{L^2}^2).$$

(ii) Let $M \in \mathcal{M}_{\text{loc}}$. Then,

$$\{[M]_\infty < \infty\} = \left\{\lim_{t\to\infty} M_t \text{ exists}\right\},$$

$$\{[M]_\infty = \infty\} = \left\{\liminf_{t\to\infty} M_t = -\infty,\ \limsup_{t\to\infty} M_t = \infty\right\}.$$

## 6.1 Exponential MGs

Let $M \in \mathcal{M}_{\text{loc}}, M_0 = 0$. Set

$$Z_t = \exp\left(M_t - \tfrac{1}{2}[M]_t\right).$$

By Itô's formula,

$$\mathrm{d}Z_t = Z_t\left(\mathrm{d}M_t - \tfrac{1}{2}d[M]_t\right) + \tfrac{1}{2}d[M]_t = Z_t\ \mathrm{d}M_t$$

giving $Z \in \mathcal{M}_{\text{loc}}, \quad Z_0 = 1$.

**Definition 6.4** (Exponential MG)**.** *In the setting above, the process $\mathcal{E}(M)_t = Z_t = \exp\left(M_t - \frac{1}{2}[M]_t\right)$ is the stochastic exponential or exponential martingale associated with $M$*

Note that $\mathcal{E}(M) \in \mathcal{M}_{\text{loc}},\ d\mathcal{E}(M)_t = \mathcal{E}(M)_t\ \mathrm{d}M_t$.

**Proposition 6.5.** *Let $M \in \mathcal{M}_{loc}, M_0 = 0$. If $[M]_\infty$ is bounded, then $\mathcal{E}(M)$ is a UI martingale.*

**Proposition 6.6.** *Let $M \in \mathcal{M}_{loc}, M_0 \geq 0$. For all $\varepsilon, \delta > 0$, we have that*

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon,\ [M]_\infty < \delta\right) \leq e^{-\frac{\varepsilon^2}{2\delta}}.$$

*Proof.* Fix $\varepsilon > 0$ and let $T = \inf\{t \geq 0 : M_t \geq \varepsilon\}$. Fix $\theta > 0$ and set $Z_t = \mathcal{E}(\theta M^T)_t$, i.e.

$$Z_t = \exp\left(\theta M_t^T - \frac{\theta^2}{2}[M^T]_t\right) \in \mathcal{M}_{\text{loc}}.$$

Note that $|Z_t| \leq e^{\theta\varepsilon}$ for all $t \geq 0$. So $Z$ is a bounded MG, hence $\mathbb{E}[Z_\infty] = Z_0 = 1$. For $\delta \geq 0$, we have that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon, [M]_\infty \leq \delta\right) &= \mathbb{P}\left(\sup_{t\geq 0} \theta M_t^T \geq \theta\varepsilon, [M^T]_\infty \leq \delta\right) \\
&\leq \mathbb{P}\left(\sup_{t\geq 0} Z_t \geq Ce^{\theta\varepsilon - \frac{\theta^2}{2}\delta}\right) \quad \text{(Doob's inequality)} \\
&\leq C\exp\left(-\theta\varepsilon + \frac{\theta^2}{2}\delta\right).
\end{aligned}$$

Optimising over $\theta$ gives the claimed bound. □

*Proof of (previous) proposition.* We will show that $\mathcal{E}(M)$ is bounded by an integrable random variable. Note that

$$\sup_{t\geq 0} \mathcal{E}(M)_t \leq \exp\left(\sup_{t\geq 0} M_t\right) \quad (\text{since } [M]_t \geq 0).$$

**NTS:** RHS is integrable. Let $C > 0$ so that $[M]_\infty \leq C$. Then:

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon\right) = \mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon,\ [M]_\infty \leq C\right) \leq \exp\left(-\frac{\varepsilon^2}{2C}\right)$$

which implies

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sup_{t\geq 0} M_t\right)\right] &= \int_0^\infty \mathbb{P}\left(\exp\left(\sup_{t\geq 0} M_t\right) \geq \lambda\right) \mathrm{d}\lambda = \int_0^\infty \mathbb{P}\left(\sup_{t\geq 0} M_t \geq \log\lambda\right) \mathrm{d}\lambda \\
&\leq 1 + \int_1^\infty \exp\left(-\frac{(\log\lambda)^2}{2C}\right) \mathrm{d}\lambda < \infty
\end{aligned}$$

finally giving that $\mathcal{E}(M)$ is UI. □

Lecture 18

Suppose that $Q, P$ are probability measures on $(\Omega, \mathcal{F})$. Say that $Q$ is absolutely continuous w.r.t. $P$, denoted by $Q \ll P$, if for any $A \in \mathcal{F}$ with

$$P(A) = 0 \Rightarrow Q(A) = 0.$$

Recall from measure theory that this implies the existence of a random variable $Z \geq 0$ such that

$$Q(A) = \mathbb{E}[Z \cdot \mathbf{1}_A] \quad \text{for all } A \in \mathcal{F}.$$

$Z$ is called the *Radon-Nikodym derivative* of $Q$ w.r.t. $P$ and is denoted by $Z = \frac{dQ}{dP}$.

**Example.** Suppose that $X \sim \mathcal{N}(0,1)$, $\mu \in \mathbb{R}$. Let

$$Z = \exp\left(\mu X - \frac{\mu^2}{2}\right).$$

Then $A \mapsto \mathbb{E}[\mathbf{1}_A Z]$ defines a probability measure $Q$, and under $Q$, $X \sim \mathcal{N}(\mu, 1)$.

The Girsanov Theorem generalizes this idea to the setting of semi-martingales, except instead of changing the mean, we will change the semi-martingale decomposition.

**Theorem 6.7** (Girsanov)**.** *Let $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$, and assume that $Z = \mathcal{E}(M)$ is uniformly integrable. Then we can construct a new probability measure $\tilde{\mathbb{P}} \ll \mathbb{P}$ on $(\mathcal{F}_t)$ by setting*

$$\tilde{\mathbb{P}}(A) := \mathbb{E}[Z_\infty \mathbf{1}_A] \quad \forall A \in \mathcal{F}.$$

*If $X \in \mathcal{M}_{c,loc}(\mathbb{P})$, then $X - [X,M] \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$.*
*'A change of measure induces a change of drift'.*

*Girsanov.* Since $Z$ is UI, hence that $Z_\infty$ exists and $Z_\infty \geq 0$ with $\mathbb{E}[Z_\infty] = 1$ and so $\tilde{\mathbb{P}}$ defines a probability measure with $\tilde{\mathbb{P}} \ll \mathbb{P}$. Suppose that $X \in \mathcal{M}_{c,\text{loc}}(\mathbb{P})$ and set

$$T_n := \inf\{t \geq 0 : |X_t - [X,M]_t| \geq n\}.$$

Since $X - [X,M]$ is continuous (starts from zero), we have that

$$\mathbb{P}(T_n \nearrow \infty) = 1 \Rightarrow \tilde{\mathbb{P}}(T_n \nearrow \infty) = 1 \quad \text{(since } \tilde{\mathbb{P}} \ll \mathbb{P}\text{)}.$$

To prove that $Y := X - [X,M] \in \mathcal{M}_{c,\text{loc}}(\tilde{\mathbb{P}})$, it suffices to show that $Y^{T_n} := X^{T_n} - [X,M]^{T_n} \in \mathcal{M}_c(\tilde{\mathbb{P}})$. In what follows, write $X, Y$ in place of $X^{T_n}, Y^{T_n}$.
Using Itô's product rule (IBP):

$$\mathrm{d}(Z_t Y_t) = Y_t \,\mathrm{d}Z_t + Z_t \,\mathrm{d}Y_t + \mathrm{d}Y_t \,\mathrm{d}Z_t.$$

Now,

$$\begin{aligned} \mathrm{d}Z_t &= Z_t \,\mathrm{d}M_t, \\ \mathrm{d}Y_t &= \mathrm{d}X_t - \mathrm{d}[X,M]_t, \\ \mathrm{d}Y_t \,\mathrm{d}Z_t &= Z_t \,\mathrm{d}[M,Y]_t = Z_t \,\mathrm{d}[X,M]_t. \end{aligned}$$

Thus,

$$d(Z_t Y_t) = Y_t Z_t \,\mathrm{d}M_t + Z_t(\mathrm{d}X_t - \mathrm{d}[X,M]_t) + Z_t \,\mathrm{d}[X,M]_t = Z_t \,\mathrm{d}X_t + Y_t Z_t \,\mathrm{d}M_t$$

giving that $ZY \in \mathcal{M}_{c,\text{loc}}(\mathbb{P})$.
Moreover, $ZY : T \leq t$ is a stopping time, and is UI for each $t > 0$, [✍] . Since $Y$ is bounded, we also have that

$$ZY \cdot \mathbf{1}_{\{T \leq t\}} \text{ is a stopping time and UI} \Rightarrow ZY \in \mathcal{M}_c(\mathbb{P}).$$

For $s \leq t$, we have that

$$\mathbb{E}[Y_t - Y_s \mid \mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s] = 0 \quad \text{(tower property)}.$$

Since $ZY \in \mathcal{M}_c(\mathbb{P})$ we finally obtain $Y \in \mathcal{M}_c(\tilde{\mathbb{P}})$. □

**Remark.** *The quadratic variation does not change when performing a change of measures,* [✍] .

**Corollary 6.1.** *Let $B$ be a standard Brownian motion under $\mathbb{P}$, $M \in \mathcal{M}_{c,loc}$, $M_0 = 0$. Suppose that*

$$Z = \mathcal{E}(M) \text{ is UI}, \quad \text{and } \mathbb{Q}(A) = \mathbb{E}[1_A Z_\infty] \text{ for all } A \in \mathcal{F}.$$

*Then $\widetilde{B} := B - [B, M]$ is a $\mathbb{Q}$-Brownian motion.*

*Proof.* Since $\widetilde{B} \in \mathcal{M}_{\text{c,loc}}(\mathbb{Q})$ by the Girsanov theorem, and $[\widetilde{B}]_t = [B - [B, M]]_t = t$, it follows from the Lévy characterisation that $\widetilde{B}$ is a $\mathbb{Q}$-Brownian motion. □ □

**Examples 6.8.** *Suppose that $B$ is a $\mathbb{P}$-Brownian motion, $\mu \in \mathbb{R}$, $T > 0$, and let $M_t = \mu B_t$, so that*

$$Z_t = \mathcal{E}(M)_t = \exp\left(\mu B_t - \mu^2 t/2\right).$$

*Then*

$$\mathbb{Q}(A) = \mathbb{E}\left[Z_T \cdot \mathbf{1}_A\right] = \mathbb{E}\left[\exp\left(\mu B_T - \mu^2 T/2\right) \mathbf{1}_A\right] \quad \forall A \in \mathcal{F}.$$

*You render under $\mathbb{P}$ that $B_t = \widetilde{B}_t + \mu t$ for $t \in [0, T]$, and $\widetilde{B}$ is a $\mathbb{Q}$-Brownian motion.*

# 7 Stochastic Differential Equations

Let $\mathbb{M}^{d\times m}(\mathbb{R})$ denote the space of $d \times m$ matrices with real entries. Suppose that

$$\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times m}(\mathbb{R}), \quad b : \mathbb{R}^d \to \mathbb{R}^d$$

are measurable functions which are bounded on compact sets. Write $\sigma(x) = (\sigma_{ij}(x))$. Consider the SDE:

$$dX_t = \sigma(X_t)\, \mathrm{d}B_t + b(X_t)\, dt \tag{$\circledast$}$$

Equivalently,

$$dX_t^i = \sum_{j=1}^m \sigma_{ij}(X_t)\, dB_t^j + b_i(X_t)\, dt.$$

A solution to $\circledast$ consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions.
- An $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B = (B^1, \ldots, B^m) \in \mathbb{R}^m$.
- An $(\mathcal{F}_t)_{t\geq 0}$-adapted continuous process $X = (X_t^1, \ldots, X_t^d) \in \mathbb{R}^d$ such that

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds.$$

When in addition $X_0 = x \in \mathbb{R}^d$, we say that $X$ is *started from $x$*.

- We say that an SDE has a <u>weak solution</u> if for all $x \in \mathbb{R}^d$, there is a solution starting from $x$.
- There is <u>uniqueness in law</u> if all solutions starting from each $x$ have the same distribution.

- There is pathwise uniqueness if, when we fix $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $B$, then any two solutions $X, X'$ with $X_0 = X_0'$ are indistinguishable:

$$\mathbb{P}(X_t = X_t' \text{ for all } t \geq 0) = 1.$$

- We say that a solution started from $x$ is a strong solution if $X$ is adapted to the filtration generated by $B$.

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**Example.** It is possible to have the existence of a weak solution and uniqueness in law without having pathwise uniqueness. Suppose that $\beta$ is a standard Brownian motion in $\mathbb{R}$ with $\beta_0 = x$. Set

$$B_t = \int_0^t \operatorname{sgn}(\beta_s)\, ds, \quad \operatorname{sgn}(x) = 1_{\{(0,\infty)\}}(x) - 1_{\{(-\infty,0]\}}(x).$$

Note that $\operatorname{sgn}(\beta_s)$ is measurable and bounded, hence the integral is well-defined. Then,

$$x + \int_0^t \operatorname{sgn}(\beta_s)\, d\beta_s = x + \int_0^t (\operatorname{sgn}(\beta_s))^2\, d\beta_s = x + \int_0^t d\beta_s = \beta_t.$$

Therefore, $\beta$ solves the SDE

$$\begin{cases} dX_t = \operatorname{sgn}(X_t)\ \mathrm{d}B_t, \\ X_0 = x. \end{cases}$$

This SDE has a weak solution. By the Lévy characterisation, any solution to this SDE is a Brownian motion (it is in $\mathcal{M}_{c,loc}$ with quadratic varitaion $[\cdot]_t = t$) which gives uniqueness in law. However, we do not have pathwise uniqueness. To see this, take $X = x = 0$.

**Claim:** $\beta_t, -\beta_t$ are solutions.

Indeed, $\beta_t$ is a solution. For $-\beta_t$, we also obtain

$$\begin{aligned} -\beta_t = -\int_0^t \operatorname{sgn}(\beta_s)\, ds &= \int_0^t \operatorname{sgn}(-\beta_s)\, d(-\beta_s) \\ &= \int_0^t \operatorname{sgn}(-\beta_s)\, dB_s + 2\int_0^t 1_{\{\beta_s = 0\}}\, dB_s. \end{aligned}$$

The last term on the RHS is in $\mathcal{M}_{c,\text{loc}}$, starts from 0, and has quadratic variation

$$4\int_0^t 1_{\{\beta_s=0\}}\, ds = 0 \quad \text{a.s.}$$

because $\mathbb{P}(\beta_s = 0) = 0\ \forall s > 0$, and then one can apply Fubini's theorem to obtain that its expectation vanishes. Therefore $\beta_t, -\beta_t$ are both solutions on the same probability space with the same Brownian motion. So we do *not* have pathwise uniqueness.

## 7.1 Lipschitz Coefficients

Recall that for $U \subset \mathbb{R}^d$ open, $f : U \to \mathbb{R}^d$, we say that $f$ is **Lipschitz** if there exists $K < \infty$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \forall x, y \in U.$$

For $d, m \geq 1$, we equip $\mathcal{M}_{d\times m}(\mathbb{R})$ with the Frobenius norm. If $A \in \mathcal{M}_{d\times m}(\mathbb{R})$, $A = (a_{ij})$, then

$$|A| = \left( \sum_{i=1}^{d} \sum_{j=1}^{m} a_{ij}^2 \right)^{1/2}.$$

Let $f : U \to \mathcal{M}_{d\times m}(\mathbb{R})$. Say that $f$ is Lipschitz if there exists $K < \infty$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \forall x, y \in U.$$

**Theorem 7.1** (Existence and Uniqueness)**.** *Suppose that*

$$\sigma : \mathbb{R}^d \to \mathcal{M}_{d\times m}(\mathbb{R}), \quad b : \mathbb{R}^d \to \mathbb{R}^d$$

*are Lipschitz. Then there is pathwise uniqueness for the SDE*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

*Moreover, for each filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and each $(\mathcal{F}_t)$-Brownian motion $B$, $x \in \mathbb{R}^d$, there is a strong solution starting from $x$.*

The proof is analogous to the existence/uniqueness theorem for ODEs. Recall some results from analysis/ODEs.

**Theorem 7.2** (Banach Fixed Point Theorem)**.** *Let $(X, d)$ be a complete metric space.*

*(a) Suppose that $F : X \to X$ is a contraction, i.e., $\exists r \in (0, 1)$ such that*

$$d(F(x), F(y)) \leq r\, d(x, y) \quad \forall x, y \in X.$$

*Then $F$ has a unique fixed point.*

*(b) Suppose that $F : X \to X$, and there exists $n \in \mathbb{N}$ so that $F^{(n)}$ is a contraction. Then $F$ has a unique fixed point.*

**Lemma 7.3** (Gronwall)**.** *Let $T > 0$ and $f : [0, T] \to [0, \infty)$ be a bounded and measurable function. If there exist $a, b > 0$ such that*

$$f(t) \leq a + b \int_0^t f(s)\, ds \quad \forall t \in [0, T],$$

*then $f(t) \leq ae^{bt}$ for all $t \in [0, T]$.*

*Proof.* [✍] . □

**Proof of Existence and Uniqueness** We will assume that dim $= 1$ and will let $K$ be such that

$$|\sigma(x) - \sigma(y)| \leq K|x - y|, \quad |b(x) - b(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}.$$

*Proof of Uniqueness.* Suppose that $X, X'$ are two solutions on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and Brownian motion $B$. WTS: $\mathbb{P}(X_t = X'_t \ \forall t \geq 0) = 1$.
Fix $M > 0$ and let

$$\tau = \inf\{t \geq 0 : |X_t| \vee |X'_t| \geq M\}.$$

Then,

$$X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X_s)\, dB_s + \int_0^{t\wedge\tau} b(X_s)\, ds,$$

$$X'_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \sigma(X'_s)\, dB_s + \int_0^{t\wedge\tau} b(X'_s)\, ds.$$

Fix $T > 0$. If $t \in [0, T]$, we have that

$$\mathbb{E}\left[(X_{t\wedge\tau} - X'_{t\wedge\tau})^2\right] \leq 2\cdot\mathbb{E}\left[\left(\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s))\, dB_s\right)^2\right] + 2\cdot\mathbb{E}\left[\left(\int_0^{t\wedge\tau} (b(X_s) - b(X'_s))\, ds\right)^2\right]$$

$$\leq 2\cdot\mathbb{E}\left[\int_0^{t\wedge\tau} (\sigma(X_s) - \sigma(X'_s))^2\, ds\right] + 2T\cdot\mathbb{E}\left[\frac{1}{T}\int_0^{t\wedge\tau} (b(X_s) - b(X'_s))^2\, ds\right] \quad \text{(Itô isometry + Cauchy–Schwarz)}$$

$$\leq 2K^2(1+T)\cdot\mathbb{E}\left[\int_0^{t\wedge\tau} |X_s - X'_s|^2\, ds\right]$$

$$= 2K^2(1+T)\int_0^t \mathbb{E}\left[|X_{s\wedge\tau} - X'_{s\wedge\tau}|^2\right] ds.$$

Let $f(t) := \mathbb{E}[|X_{t\wedge\tau} - X'_{t\wedge\tau}|^2]$. Then:

$$0 \leq f(t) \in 4M^2 \text{ and } f(t) \leq 2K^2(1+T)\int_0^t f(s)\, ds \quad \forall t \in [0, T].$$

By Gronwall's inequality, $f(t) = 0$ for all $t \in [0, T]$, so

$$\mathbb{P}(X_{t\wedge\tau} = X'_{t\wedge\tau} \ \forall t \in [0, T]) = 1.$$

Since $M, T$ were arbitrary, we conclude:

$$\mathbb{P}(X_t = X'_t \ \forall t \geq 0) = 1.$$

That is, we have established **Pathwise uniqueness**. □

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*Proof of existence.* Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space, $B$ is an $(\mathcal{F}_t)$-Brownian motion, and $(\mathcal{F}_t^B)_{t\geq 0}$ is the filtration generated by $B$ (so that $\mathcal{F}_t^B \subseteq \mathcal{F}_t$). We will use the contraction mapping theorem. Need to specify

1) the space,

2) the map.

For each $T > 0$, let $\mathcal{C}_T = \{\text{continuous, adapted processes } X : [0, T] \to \mathbb{R}\}$, with

$$\|X\|_T := \left(\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right]\right)^{1/2}.$$

We proved before that $\mathcal{C}_T$ is complete. Fix $x \in \mathbb{R}$. Using that $\sigma, b$ are Lipschitz, we have

$$|\sigma(y)| = |\sigma(y) - \sigma(0) + \sigma(0)| \le |\sigma(y) - \sigma(0)| + |\sigma(0)| \le K|y| + |\sigma(0)|, \qquad ((1))$$

$$|b(y)| \le |b(0)| + K|y| \quad \text{for all } y \in \mathbb{R}. \qquad ((2))$$

Fix $T > 0$, and $X \in \mathcal{C}_T$. Let

$$M_t := \int_0^t \sigma(X_s)\, dB_s, \quad 0 \le t \le T.$$

Then,

$$[M]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Thus, by (1),

$$\mathbb{E}[[M]_T] \le 2T \left( |\sigma(0)|^2 + K^2 \|X\|_T^2 \right) < \infty.$$

which implies that $M \in \mathcal{M}_c^2$, so by Doob's inequality,

$$\mathbb{E}\left[ \sup_{0 \le t \le T} \left| \int_0^t \sigma(X_s)\, dB_s \right|^2 \right] \le 8T \left( |\sigma(0)|^2 + K^2 \|X\|_T^2 \right).$$

By (2),

$$\mathbb{E}\left[ \sup_{0 \le t \le T} \left| \int_0^t b(X_s)\, ds \right|^2 \right] \le \dots$$

$$\le T \cdot \mathbb{E}\left[ \int_0^T b(X_s)^2\, ds \right] \quad \text{(Cauchy-Schwarz)}$$

$$\le 2T \cdot \mathbb{E}\left[ |\sigma(0)|^2 + K^2 \|X\|_T^2 \right] < \infty$$

The map $F$ on $\mathcal{C}_T$ defined by

$$F(X)_t = x + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds$$

takes values in $\mathcal{C}_T$.
Suppose that $X, Y \in \mathcal{C}_T$. For $0 \le t \le T$, using similar arguments,

$$\|F(X) - F(Y)\|_t^2 \le 4K^2 T \cdot (4 + T) \int_0^t \|X - Y\|_s^2\, ds = C_T \int_0^t \|X - Y\|_s^2\, ds$$

Iterate $n$ times:

$$\left\| F^{(n)}(X) - F^{(n)}(Y) \right\|_T^2 \le C_T^n \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|X - Y\|_T^2\, dt_n \cdots dt_1$$

$$\le \frac{C_T^n T^n}{n!} \|X - Y\|_T^2 \qquad (3)$$

Take $n$ sufficiently large so that $\frac{C_T^n T^n}{n!} < 1$. Then by the contraction mapping theorem, there exists a unique fixed point $X^{(T)} \in \mathcal{C}_T$ of $F$. Pathwise uniqueness $\Rightarrow$ $X_t^{(T)} = X_t^{(T')}$ for all $t \le T \wedge T'$ a.s. Define $X_t$ by setting $X_t = X_t^{(N)}$ where $t \le N$, $N \in \mathbb{N}$. Then $X$ is the pathwise unique solution to the SDE starting from $x$.

**NTS:** $X$ is a strong solution, i.e. $X$ is adapted to $(\mathcal{F}_t^B)$. We will prove first that for each fixed $T$, $X^{(T)}$ is the limit of $(\mathcal{F}_t^B)$-processes. Define $y^0 = x$ and $y^n = F(y^{n-1})$ for each $n \in \mathbb{N}$. Then $(y^n)$ is adapted to $(\mathcal{F}_t^B)$ for each $n$. As $F^{(n)}(X) = X$, for all $n \geq d$, we have from (3) that:

$$\|X - y^n\|_T^2 = \|F^{(n)}(X) - F^{(n)}(x)\|_T^2 \leq \frac{C_T^n T^n}{n!}\|X - x\|_T^2 \to 0 \quad \text{as } n \to \infty.$$

Thus $Y^n \to X$ in $C_T$ as $n \to \infty$. So there exists a subsequence $(Y^{n_k})$ such that $Y^{n_k} \to X$ uniformly in $[0,T]$ a.s. Therefore, $(X_t)$ is the a.s. limit of $(\mathcal{F}_t^B)$-adapted processes and so is $(\mathcal{F}_t^B)$-adapted. Since $T > 0$ was arbitrary, we have that $X$ is $(\mathcal{F}_t^B)$-adapted. □

**Remark.** *From the abvoe proof, we also obtain that the pathwise unique strong solution lies in $C_T$ for all $T > 0$.*

**Proposition 7.4.** *Under the hypotheses of the theorem, there is uniqueness in law for the SDE*

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

*Proof.* [✍] . □

**Example.**(Ornstein–Uhlenbeck process) Fix $\lambda \in \mathbb{R}$ and consider the SDE

$$dV_t = dB_t - \lambda V_t dt, \quad V_0 = v_0,$$

$$dX_t = V_t dt.$$

For $\lambda > 0$, this models the movement of a grain of pollen in liquid; $X$ = position of the grain, $V$ = velocity. The term $-\lambda V$ damps the system due to viscosity. When $|V|$ is large, the system moves to reduce $|V|$.

The previous theorem implies that there exists a unique strong solution. We can explicitly solve

$$d(e^{\lambda t}V_t) = e^{\lambda t}dV_t + \lambda e^{\lambda t}V_t dt = e^{\lambda t}dB_t.$$

Hence,

$$e^{\lambda t}V_t = v_0 + \int_0^t e^{\lambda s}dB_s,$$

so that

$$V_t = e^{-\lambda t}v_0 + \int_0^t e^{-\lambda(t-s)}dB_s.$$

Therefore,

$$V_t \sim \mathcal{N}\left(e^{-\lambda t}v_0, \frac{1 - e^{-2\lambda t}}{2\lambda}\right).$$

If $\lambda > 0$, then $V_t$ converges in distribution to $\mathcal{N}\left(0, (2\lambda)^{-1}\right)$ as $t \to \infty$. Hence, $\mathcal{N}(0, (2\lambda)^{-1})$ is the stationary distribution of $V$, i.e. if $V_0 \sim \mathcal{N}(0, (2\lambda)^{-1})$, then

$$V_t \sim \mathcal{N}(0, (2\lambda)^{-1}) \quad \text{for all } t \geq 0.$$

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## 7.2 Local solutions

Consider the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt.$$

A locally defined process is a pair $(X, \tau)$ consisting of a stopping time $\tau$ together with a map

$$X : \{(\omega, t) \in \Omega \times [0, \infty) : t < \tau(\omega)\} \to \mathbb{R}.$$

It is said to be càdlàg if the map $t \mapsto X_t(\omega)$ from $[0, \tau(\omega))$ to $\mathbb{R}$ is càdlàg for all $\omega \in \Omega$. Let $\Omega_t = \{\omega \in \Omega : t < \tau(\omega)\}$. Then $(X, \tau)$ is *adapted* if $X_t : \Omega_t \to \mathbb{R}$ is $\mathcal{F}_t$-measurable. We say that $(X, \tau)$ is a locally defined martingale if there exist stopping times $\tau_n \nearrow \tau$ such that $X^{\tau_n}$ is a martingale for all $n$. We say that $(H, \eta)$ is a locally defined, locally bounded, predictable process if there exist stopping times $S_n \nearrow \eta$ such that $H\mathbf{1}_{\{0 \le t \le S_n\}}$ is bounded and predictable for all $n \in \mathbb{N}$. We define $(H \cdot X, \tau \wedge \eta)$

$$(H \cdot X)_t^{T_n \wedge S_n} = (H\mathbf{1}_{(0, S_n \wedge T_n]} \cdot X)_t \quad \text{for each } n.$$

**Proposition 7.5** (Local Itô's formula)**.** *Let $X^1, \ldots, X^d$ be continuous semimartingales, let $U \subseteq \mathbb{R}^d$ be open, and let $f : U \to \mathbb{R}$ be $C^2$. Let $X = (X^1, \ldots, X^d)$ and set*

$$\tau = \inf\{t \ge 0 : X_t \notin U\}.$$

*Then for all $t < \tau$, we have that*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d[X^i, X^j]_s.$$

*Proof.* Apply Itô's formula to $X^{\tau_n}$, where

$$\tau_n = \inf\left\{t \ge 0 : \operatorname{dist}(X_t, U^c) \le \frac{1}{n}\right\},$$

and note that $\tau_n \nearrow \tau$ as $n \to \infty$. □

**Examples 7.6.** *Let $X = B$, where $B$ is a standard Brownian motion with $X_0 = B_0 = 1$, $U = (0, \infty)$, and $f(x) = \sqrt{x}$. Then*

$$\sqrt{B_t} = 1 + \frac{1}{2} \int_0^t B_s^{-1/2}\, dB_s - \frac{1}{8} \int_0^t B_s^{-3/2}\, ds \quad \textit{for all } t < \tau,$$

*where*

$$\tau = \inf\{t \ge 0 : B_t = 0\}.$$

Let $U \subseteq \mathbb{R}^d$ be open, $\sigma : U \to \mathbb{M}^{d \times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ be measurable functions which are bounded on compact subsets of $U$.

A local solution to the SDE

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions.
- An $(\mathcal{F}_t)$-Brownian motion $B$ in $\mathbb{R}^m$.
- A continuous $(\mathcal{F}_t)$-adapted locally defined process $(X, \tau)$, with $X \in \mathbb{R}^d$, such that

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds, \quad \text{for all } t < \tau.$$

We say that $(X, \tau)$ is a *maximal local solution* if for any other local solution $(\tilde{X}, \eta)$ on the same space such that

$$X_t = \tilde{X}_t \quad \text{for all } t \leq \tau \wedge \eta,$$

we have that $\eta \leq \tau$.

**Locally Lipschitz coefficients:** Suppose that $U \subseteq \mathbb{R}^d$ is open. Then a function $f : U \to \mathbb{R}^d$ is locally Lipschitz if for each compact set $C \subseteq U$, we have that $f|_C$ is Lipschitz.

**Theorem 7.7.** *Suppose $U \subseteq \mathbb{R}^d$ is open and $\sigma : U \to \mathbb{M}^{d\times m}(\mathbb{R})$, $b : U \to \mathbb{R}^d$ are locally Lipschitz. Then for all $x \in U$, the SDE*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

*has a pathwise unique maximal local solution $(X, \tau)$ starting from $x$. Moreover, for all compact sets $C \subseteq U$, on the event that $\tau < \infty$, we have that*

$$\sup\{t < \tau : X_t \in C\} < \tau.$$

**Lemma 7.8.** *Let $U \subseteq \mathbb{R}^d$ be open, $C \subseteq U$ be compact. Then:*

1. *There exists a $C^\infty$ function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi|_C \equiv 1$ and $\varphi|_{U^c} \equiv 0$.*
2. *Given a locally Lipschitz function $f : U \to \mathbb{R}$, then there exists a globally Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ such that $f|_C = g|_C$.*

*Proof:* (i) .

(ii) Let $\varphi$ be as in part (i) and set $g = f \cdot \varphi$. □

□

*Proof (Theorem).* Assume that $d = m = 1$. Fix $C \subseteq U$ compact. By the lemma, we can find Lipschitz functions $\tilde{\sigma}, \tilde{b}$ on $\mathbb{R}$ such that $\tilde{\sigma}|_C = \sigma|_C$, $\tilde{b}|_C = b|_C$. Then there exists a pathwise unique strong solution $\tilde{X}$ to:

$$\begin{cases} d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t) dB_t + \tilde{b}(\tilde{X}_t) dt \\ \tilde{X}_0 = x \end{cases}$$

Let $\tau = \inf\{t \geq 0 : \tilde{X}_t \notin C\}$ and let $X = \tilde{X}|_{[0,\tau)}$. Then $(X, \tau)$ is a local solution in $C$, . If $\tau < \infty$, then $X_{\tau^-} = \lim_{t\to\tau^-} X_t$ exists and is in $U^C$. Suppose that $(X, \tau), (Y, \eta)$ are both local solutions in $C$. Let

$$f(t) = \mathbb{E}\left[\sup_{0\le s\le t\wedge\tau\wedge\eta} |X_s - Y_s|^2\right]$$

As $b, \sigma$ are Lipschitz on $C$, we can use Gronwall's lemma as before to see that $f \equiv 0$, which implies that $X_t = Y_t$ for all $t \le \eta \wedge \tau$ almost surely.

Let $(C_n)$ be a sequence of compact sets in $U$ with $C_n \subseteq C_{n+1}$ for all $n$, and $U = \bigcup_n C_n$. Let $(X^n, T_n)$ be the local solution constructed above with $C = C_n$. If $T_n < \infty$, then $X^n_{T_n} \in U \setminus C_n^\circ$. Observe that on

$$\underbrace{\inf\{t \ge 0 : X_t^{n+1} \notin C_n^\circ\}}_{:=\tilde{T}_n} \wedge T_n := S_n$$

we have

$$X_t^{n+1} = X_t^n \quad \text{almost surely for all } t \le S_n$$

(by a Gronwall-type argument). Suppose for a contradiction that $\tilde{T}_n < T_n$. Then the above implies

$$X_{\tilde{T}_n}^{n+1} = X_{\tilde{T}_n}^n \quad \text{almost surely,} \quad \text{and } t \le \tilde{T}_n$$

giving

$$X_{\tilde{T}_n}^{n+1} = X_{\tilde{T}_n}^n \notin C_n^\circ \subseteq C_n$$

Hence

$$T_n \le \tilde{T}_n \le T_{n+1} \quad \text{which implies that } (T_n) \text{ is increasing.}$$

Since the $T_n$ are non-decreasing, we have $T_n \nearrow \tau$, i.e., $\tau = \sup_n T_n$.

Define the local solution by setting $X_t = X_t^n$ for all $t < T_n$. This is consistent by the above. We now aim to show that $(X, \tau)$ is maximal.

Lecture 22 It thus remains to show

1. maximality,
2. $\sup\{t < \tau : X_t \in C\} < \tau$ on the event $\{\tau < \infty\}$.

Suppose that $(Y, \eta)$ is another solution on the same probability space. For each $n$, set

$$S_n = \inf\{t \in [0, \infty) : Y_t \notin C_n\} \wedge \eta.$$

By the uniqueness of the solution in each $C_n$, we have that $X_t = Y_t$ for all $t \le S_n \wedge T_n$. Therefore, arguing as before, $S_n \le T_n$. As $n \to \infty$, $S_n \nearrow \eta$, $T_n \nearrow \tau$, so

$$\eta \le \tau, \quad X_t = Y_t \text{ for all } t \le \eta.$$

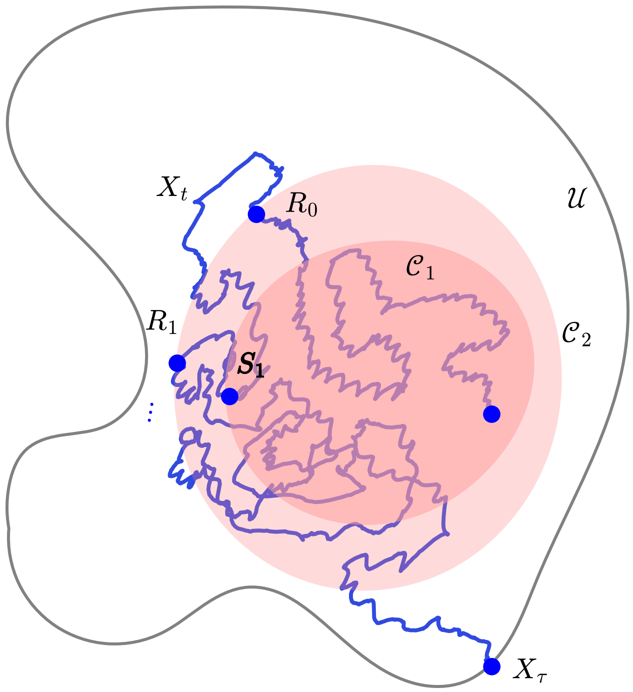
Therefore, $(X, \tau)$ is maximal. □

Suppose that $C_1, C_2$ are compact sets in $\mathcal{U}$ with $C_1 \subseteq C_2^\circ \subseteq C_2 \subseteq \mathcal{U}$. Let $\varphi : \mathcal{U} \to \mathbb{R}$ be a $C^\infty$ function with $\varphi|_{C_1} \equiv 1$, $\varphi|_{(C_2^\circ)^c} \equiv 0$. Let

$$R_0 = \inf\{t \ge 0 : X_t \notin C_2\},$$

$$S_n = \inf\{t \ge R_{n-1} : X_t \notin C_1\} \wedge \tau,$$

$$R_n = \inf\{t \ge S_n : X_t \notin C_2\} \wedge \tau.$$




Let $N$ be the number of crossings that $X$ makes from $C_2$ to $C_1$. On the event $\{\tau \leq t,\, N \geq n\}$, we have that:

$$\sum_{k=1}^{n} \left(\varphi(X_{R_k}) - \varphi(X_{S_k})\right) = -n$$

$$= \int_0^t \sum_{k=1}^{n} \mathbf{1}_{(S_k, R_k]}(s) \left(\varphi(X_s)\, dX_s + \frac{1}{2}\varphi''(X_s)\, d[X]_s\right)$$

$$= \int_0^t H_s^n\, dB_s + K_s^n\, ds =: Z_t^n,$$

where $H^n$, $K^n$ are predictable and bounded uniformly in $n$. Then:

$$n \cdot \mathbf{1}_{\{\tau \leq t,\, N \geq n\}} \leq (Z_t^n)^2 \Rightarrow \mathbb{P}\left(\tau \leq t,\, N \geq n\right) \leq \frac{1}{n^2}\mathbb{E}\left[(Z_t^n)^2\right]$$

Since $H^n$, $K^n$ are uniformly bounded and $Z_t^n$ is defined by integrating $H^n$, $K^n$ over a time-interval which does not depend on $n$, we have that

$$\mathbb{E}\left[(Z_t^n)^2\right] \leq C \text{ where } C \text{ does not depend on } n \Rightarrow \mathbb{P}\left(\tau \leq t,\, N \geq n\right) \leq \frac{C}{n^2}.$$

Letting $n \to \infty$ gives

$$\mathbb{P}\left(\tau \leq t,\, N = \infty\right) = 0 \Rightarrow \mathbb{P}\left(\tau < \infty,\, N = \infty\right) = 0$$

Therefore, the number of crossings that $X$ makes from $C_2$ to $C_1$ is finite on the event $\{\tau < \infty\}$ almost surely.

**Example.** (Bessel processes) Fix $v \in \mathbb{R}$ and consider the SDE in $U = (0, \infty)$ given by:

$$dX_t = dB_t + \frac{n-1}{2X_t}\,dt, \quad X_0 = x_0 \in U.$$

Then there exists a unique maximal local solution $(X, \tau)$ in $U$ and $M_t := \mathbb{P}\left[\exists t \geq 0 : X_t = 0\right] = 0$. $(X, \tau)$ is a **Bessel process of dimension** $n$.

Suppose that $n \in \mathbb{N}$, $\beta$ is a Brownian motion in $\mathbb{R}^n$ with $|\beta_0| = x_0 > 0$. Set $X_t := |\beta_t|$ and

$$\tau := \inf\left\{t \geq 0 : \beta_t = 0\right\}.$$

By the local Itô formula, we have that

$$dX_t = (\beta_t, d\beta_t) + \frac{n-1}{2|\beta_t|}dt, \quad t < \tau,$$

where $(\cdot, \cdot)$ is the Euclidean inner product. Then the process

$$W_t := \int_0^t \frac{(\beta_s, d\beta_s)}{|\beta_s|} \quad \text{is a local martingale.}$$

Moreover,

$$d[W]_t = \frac{1}{|\beta_t|^2} \sum_{i,j=1}^{n} \beta_t^i \beta_t^j \, d[\beta^i, \beta^j]_t = dt.$$

Lévy's characterization implies that $W$ is a standard Brownian motion. Hence,

$$dX_t = dW_t + \frac{n-1}{2X_t}dt, \quad t < \tau.$$

A Bessel process of dimension $v$ describes the true evolution of the norm of an $v$-dimensional Brownian motion up to when it first hits 0.

## 7.3 Diffusion Processes

Suppose that $a : \mathbb{R}^d \to \mathbb{M}^{d\times d}(\mathbb{R})$, $b : \mathbb{R}^d \to \mathbb{R}^d$ are bounded, measurable, $a$ is symmetric (i.e., $a(x)$ is symmetric for each $x$). For $f \in C_b^2(\mathbb{R}^d)$ (i.e., $C_b^2$ with bounded derivatives), set

$$Lf(x) := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i}.$$

Let $X$ be a continuous, adapted process in $\mathbb{R}^d$. We say that $X$ is an $L$-diffusion if for all $f \in C_b^2(\mathbb{R}^d)$ we have that:

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds \quad \text{is a martingale.}$$

(The coefficient $a$ is called the diffusion, and $b$ is the drift.)

**Example.** $\sigma$, $b$ constant and $a = \sigma\sigma^\top$. $B$ is standard BM on $\mathbb{R}^d$. Then

$$X_t = \sigma B_t + bt \quad \text{is an } (\sigma, b)\text{-diffusion.}$$

If $\sigma = I_d$, $b = 0$, $X_t = B_t$ is an $L$-diffusion where $L = \frac{1}{2}\Delta$.

**Proposition 7.9.** *Suppose that $X$ solves*

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,$$

*let $f \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ (bounded derivatives, $C^1$ in the first variable, $C^2$ in the second variable).*
*Then,*

$$M_t^f := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + L)\, f(s, X_s)\, ds \quad \text{is a martingale,}$$

*$a = \sigma\sigma^\top$ and $L$ as above.*
*If $a$, $b$ are bounded, then $X$ is an $L$-diffusion.*

*Proof.* [✍] . □

Lecture 23 **Question:** Which $a$ can be written as $\sigma\sigma^\top$ for such $\sigma$? (See proposition from last time.)
Suppose that $a$, $b$ are Lipschitz, bounded, and there exists $\varepsilon > 0$ so that:

$$(a(x)\xi, \xi) \geq \varepsilon|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d.$$

Then $a$ is uniformly positive definite (UPD). Then there exists $\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times d}(\mathbb{R})$ with $\sigma\sigma^\top = a$.

For $d = 1$, take $\sigma = \sqrt{a}$.

For $d \geq 2$, we can write $a(x) = U(x)\Lambda(x)U(x)^\top$ where $\Lambda(x)$ is the diagonal matrix of eigenvalues and $U(x)$ the orthogonal matrix whose columns are eigenvectors of $a(x)$. Take

$$\sigma(x) = U(x)\sqrt{\Lambda(x)}U(x)^\top.$$

That $\sigma$ is Lipschitz follows from the differentiability of the square root map on the set of UPD matrices.

For such $\sigma$, $b$, the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

has a unique strong solution which is an $(a, b)$-diffusion.

**Proposition 7.10.** *Let $X$ be an $L$-diffusion and $\tau$ a finite stopping time. Set*

$$\tilde{X}_t = X_{\tau+t}, \quad \text{and} \quad \tilde{\mathcal{F}}_t = \mathcal{F}_{\tau+t}.$$

*Then $\tilde{X}$ is an $L$-diffusion with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.*

*Proof.* Fix $f \in C_0^2(\mathbb{R}^d)$. Consider the process

$$\tilde{M}_t^f := f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t Lf(\tilde{X}_s)ds.$$

$\tilde{M}_t^f$ is adapted to $(\tilde{\mathcal{F}}_t)$ and is integrable. For $A \in \mathcal{F}_s$ and $n \geq 0$ we have that

$$\mathbb{E}\left[(\tilde{M}_t^f - \tilde{M}_s^f) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}}\right] = \mathbb{E}\left[(M_{t+\tau}^f - M_{s+\tau}^f) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}}\right]$$

$$= \mathbb{E}\left[(M^f_{t+\tau} - M^f_{s+\tau}) \cdot \mathbf{1}_{A\cap\{\tau\leq n\}\in\mathcal{F}_{\tau+s}}\right]$$
$$= 0 \quad \text{(by optional stopping theorem).}$$

Sending $n \to \infty$ implies

$$\mathbb{E}\left[(\tilde{M}^f_t - \tilde{M}^f_s) \cdot \mathbf{1}_A\right] = 0 \quad \text{(by dominated convergence theorem).}$$

So $\tilde{M}^f$ is a martingale with respect to $(\tilde{\mathcal{F}}_t)$. □

**Lemma 7.11.** *Let $X$ be an $L$-diffusion. Then for all $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ the process*

$$M^f_t = f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + L) f(s, X_s) ds$$

*is a martingale.*

*Proof.* Fix $T > 0$ and consider

$$Z_n = \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq 1/n}} \left|\dot{f}(s, X_t) - \dot{f}(s, X_s)\right| + \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq 1/n}} |Lf(s, X_t) - Lf(t, X_t)|\,.$$

Then $Z_n$ is bounded and $Z_n \to 0$ as $n \to \infty$ by continuity. By the bounded convergence theorem, it follows that

$$\mathbb{E}[Z_n] \to 0 \quad \text{as } n \to \infty.$$

Now,

$$\begin{aligned} M^f_t - M^f_s = &\left(f(t, X_t) - f(s, X_t) - \int_s^t \dot{f}(r, X_t)\, dr\right) \\ &+ \left(f(s, X_t) - f(s, X_s) - \int_s^t Lf(s, X_r)\, dr\right) \\ &+ \left(\int_s^t \dot{f}(r, X_t) - \dot{f}(r, X_r)\, dr\right) \\ &+ \left(\int_s^t Lf(s, X_r) - Lf(r, X_r)\, dr\right). \end{aligned}$$

Choose $s = s_0 < s_1 < \cdots < s_n = t$ such that $s_{k+1} - s_k \leq 1/n$ for each $k$. The first line is equal to 0 by the fundamental theorem of calculus. The second line has expectation equal to 0 given $\mathcal{F}_s$ (since $X$ is an $L$-diffusion). For the last two lines, we have that

$$\mathbb{E}\left[\left|\mathbb{E}\left[M^f_t - M^f_s \mid \mathcal{F}_s\right]\right|\right] \leq (t-s) \cdot \mathbb{E}[Z_n].$$

So,

$$\mathbb{E}\left[\mathbb{E}\left[M^f_t - M^f_s \mid \mathcal{F}_s\right]\right] \leq (t-s) \cdot \mathbb{E}[Z_n] \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\mathbb{E}\left[M^f_t \mid \mathcal{F}_s\right] = M^f_s.$$

□

## 7.4 Dirichlet and Cauchy problem

Assume that $a, b$ are Lipschitz and $a(x)\xi \cdot \xi \geq \varepsilon|\xi|^2$ for some $\varepsilon > 0$, for all $x, \xi \in \mathbb{R}^d$ (i.e., $a$ is uniformly positive definite).

Let $\mathcal{D} \subseteq \mathbb{R}^d$ be a bounded, open domain with smooth boundary. We shall assume the following theorem from PDE.

**Theorem 7.12** (Dirichlet Problem)**.** *For all $f \in C(\partial\mathcal{D})$, there exists a unique function $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ such that:*

$$\begin{cases} Lu + \varphi = 0 & \text{in } \mathcal{D}, \\ u = f & \text{on } \partial\mathcal{D}. \end{cases}$$

*Moreover, there exist continuous functions*

$$m : \mathcal{D} \times \partial\mathcal{D} \to [0, \infty), \quad g : \{(x, y) \in \mathcal{D} \times \mathcal{D} : x \neq y\} \to (0, \infty)$$

*such that for all $f, \varphi$ as above, we have*

$$u(x) = \int_{\mathcal{D}} g(x, y)\varphi(y)\, dy + \int_{\partial\mathcal{D}} f(y)\, m(x, y)\, \lambda(dy),$$

*where $g$ is the Green kernel, and $m(x, y)\, \lambda(dy)$ is the harmonic measure on $\partial\mathcal{D}$ as seen from $x$.*

**Theorem 7.13.** *Suppose that $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ satisfies*

$$\begin{cases} Lu + \varphi = 0 & \text{on } \mathcal{D}, \\ u = f & \text{on } \partial\mathcal{D}, \end{cases}$$

*with $f \in C(\partial\mathcal{D}), \varphi \in C(\overline{\mathcal{D}})$. Then for any $L$-diffusion $X$ starting from $x \in \mathcal{D}$, we have*

$$u(x) = \mathbb{E}_x \left[ \int_0^\tau \varphi(X_s)\, ds + f(X_\tau) \right],$$

*where $\tau = \inf\{t \geq 0 : X_t \notin \mathcal{D}\}$. Moreover, for all Borel sets $A \subseteq \mathcal{D}, B \subseteq \partial\mathcal{D}$, we have*

$$\mathbb{E}_x \left[ \int_0^\tau \mathbf{1}(X_s \in A)\, ds \right] = \int_A g(x, y)\, dy, \quad \mathbb{P}_x \left[ X_\tau \in B \right] = \int_B m(x, y)\, \lambda(dy).$$

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*Proof.* Fix $n \geq 1$ and let $T_n = \inf\{t \geq 0 : X_t \notin \mathcal{D}_n\}$, where $\mathcal{D}_n = \{x \in \mathcal{D} : \text{dist}(x, \mathcal{D}^c) > 1/n\}$. Consider

$$M_t = u(X_{t \wedge T_n}) - u(X_0) + \int_0^{t \wedge T_n} \varphi(X_s)\, ds.$$

There exists $\tilde{u} \in C_b^2(\mathbb{R}^d)$ with $\tilde{u} = u$ on $\mathcal{D}_n$. Then $M = \tilde{M}^{T_n}$ where:

$$\tilde{M}_t = \tilde{u}(X_t) - \tilde{u}(X_0) - \int_0^t L\tilde{u}(X_s)\, ds.$$

Since $X$ is an $L$-diffusion, $\tilde{M}$ is a martingale. By the optional stopping theorem, $M$ is a martingale. Hence,

$$u(x) = \mathbb{E}_x \left[ u(X_{T_n}) + \int_0^{T_n} \varphi(X_s)\, ds \right]. \tag{$\bigstar$}$$

We want to send $n \to \infty$. First we will show $\mathbb{E}_x[T] < \infty$. Take $\varphi \equiv 1$, $f \equiv 0$, and let $u^{1,0}$ be the solution of the associated Dirichlet problem. Then ($\bigstar$) holds for $u^{1,0}$, so:

$$\mathbb{E}_x \left[ T_n \wedge t \right] = u^{1,0}(x) - \mathbb{E}_x \left[ u^{1,0}(X_{T_n}) \right].$$

Since $u^{1,0}$ is bounded (in $C(\overline{\mathcal{D}})$), $T_n \uparrow T$ as $n \to \infty$, monotone convergence theorem implies $\mathbb{E}_x[T] < \infty$ (as $n \to \infty$, $t \to \infty$).

Now return to the general case in ($\bigstar$). Have that $T_n \wedge t \nearrow T$ as $n, t \to \infty$. Since $u$ is continuous on $\overline{\mathcal{D}}$,

$$u(X_{t \wedge T_n}) \to f(X_T) \quad \text{as } n, t \to \infty.$$

Since $u$ is bounded on $\overline{\mathcal{D}}$ ($\overline{\mathcal{D}}$ compact, $u$ continuous), bounded convergence theorem implies

$$\mathbb{E}_x \left[ u(X_{t \wedge T_n}) \right] \to \mathbb{E}_x \left[ f(X_T) \right] \quad \text{as } t, n \to \infty.$$

Moreover,

$$\mathbb{E}_x \left[ \int_0^T |\varphi(X_s)|\, ds \right] \leq \|\varphi\|_\infty \cdot \mathbb{E}_x[T] < \infty.$$

By the dominated convergence theorem,

$$\mathbb{E}_x \left[ \int_0^{T \wedge t \wedge T_n} \varphi(X_s)\, ds \right] \to \mathbb{E}_x \left[ \int_0^T \varphi(X_s)\, ds \right].$$

Thus,

$$u(x) = \mathbb{E}_x \left[ f(X_T) + \int_0^T \varphi(X_s)\, ds \right].$$

Final assertions follow by taking limits as $\varphi_n \to \mathbf{1}_A$, $f \equiv 0$ and $f_n \to \mathbf{1}_B$, $\varphi \equiv 0$. □

**Theorem 7.14.** *For each $f \in C_b^2$, there exists a unique solution $u \in C_b^1(\mathbb{R}_+ \times \mathbb{R}^d)$ such that:*

$$\begin{cases} \partial u / \partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0, x) = f & \text{on } \mathbb{R}^d \end{cases}$$

*Moreover, there exists a continuous function $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ such that*

$$u(t, x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy \qquad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

*where $p$ is the "heat kernel".*

**Theorem 7.15.** *Assume that $f \in C_b^2(\mathbb{R}^d)$. Let $u$ satisfy*

$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,x) = f & \text{on } \mathbb{R}^d \end{cases}$$

*Then for any $L$-diffusion $X$ starting from $x$, for all $t \in \mathbb{R}_+$, $0 \le s \le t$, we have that*

$$\mathbb{E}_x\left[f(X_t)|\mathcal{F}_s\right] = u(t-s, X_s) \quad \text{almost surely.}$$

*In particular,*

$$\mathbb{E}_x\left[f(X_t)\right] = u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy.$$

*Finally, under $\mathbb{P}_x$, the finite-dimensional distributions of $X$ are given by:*

$$\mathbb{P}_x\left[X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\right] = p(t_1, x_0, x_1) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n)\,dx_1 \cdots dx_n,$$

*for $0 < t_1 < t_2 < \cdots < t_n < \infty$, $x_1, \dots, x_n \in \mathbb{R}^d$, $x_0 = x$.*

*Proof.* Fix $t \in (0,\infty)$. Consider $g(s,x) = u(t-s,x)$ for $s \le t$, $x \in \mathbb{R}^d$. Note that

$$\left(\frac{\partial}{\partial s} + L\right) g(s,x) = -\frac{\partial u}{\partial t}(t-s,x) + Lu(t-s,x) = 0.$$

Therefore,

$$M_s^g = g(s,X_s) - g(0,X_0) - \int_0^s \left(\frac{\partial}{\partial r} + L\right) g(r,X_r)\,dr = g(s,X_s) - g(0,X_0)$$

is a martingale for $s \in [0,t)$. By extending $g$ to $\tilde{g} \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ appropriately. Hence, for all $0 \le s \le t' < t$, we have

$$\mathbb{E}_x\left[M_{t'}^g|\mathcal{F}_s\right] = M_s^g \text{ almost surely}, \quad \Rightarrow \quad \mathbb{E}_x[M_{t'}^g] = \mathbb{E}_x[M_0^g].$$

Therefore,

$$\mathbb{E}_x[u(t-t', X_{t'})] = u(t,x).$$

Now, as $t' \to t$, by continuity $u(t-t', X_{t'}) \to f(X_t)$ (bounded convergence, $u \in C_b^2$), so

$$\mathbb{E}_x[f(X_t)] = u(t,x).$$

For the second part of the theorem set

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy = u(t,x).$$

**Uniqueness of solutions to the Cauchy problem:**

$$P_s(P_t f) = P_{s+t} f$$

**Claim (by induction):**

$$\mathbb{E}_x\left[\prod_{i=1}^n f_i(X_{t_i})\right] = \int_{(\mathbb{R}^d)^n} p(t_1,x_0,x_1)f_1(x_1)\cdots p(t_n - t_{n-1}, x_{n-1}, x_n) f_n(x_n)\,dx_1\cdots dx_n$$

**For induction, we use that:**

$$\mathbb{E}_{x_0}\left[\prod_{i=1}^{n} f_i(X_{t_i})\right] = \prod_{i=1}^{n-1} f_i(X_{t_i})\,\mathbb{E}\left[f_n(X_{t_n}) \mid \mathcal{F}_{t_{n-1}}\right]$$
$$= \prod_{i=1}^{n-1} f_i(X_{t_i})\,P_{t_n - t_{n-1}} f(X_{t_{n-1}})$$

Now apply the case $n-1$. $\square$