Random walks and phase transitions based on course by P. F. Rodriguez

Notes taken by Pantelis Tassopoulos

Lent 2026

These notes are produced entirely from the Part III course with the same title in Lent 2026, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. All errors are almost surely mine. Please send any corrections to pkt28@cam.ac.uk.

Contents

1 Introduction

The purpose of these lectures is two-fold. First, we will discuss " random walks on weighted graphs". These are quite general reversible Markov chains $X = (X_n)_{n\geq 0}$ on countable state spaces G. One focus will be on the potential theory of X.

We will then introduce the Gaussian free field associated to X. This is a canonical random field $\varphi : G \to \mathbb{R}$ attached to this setup, that plays a foundational role in (scalar) field theory, and also an increasingly prominent role in probability.

In the second part, we will study a phase transition relaed to the topological properties of the sets $\{x \in G : \varphi_x \geq h\}$ as the height parameter $h \in \mathbb{R}$ varies. This will create a bridge to the world of percolation, and the study of this transition benefits from the rich mathematical structure underlying φ (and X).

2 Random walks and potential theory

Our typical setup will be the following. We consider $\Gamma = (G, \mu, \kappa)$ a **weighted graph**; that is G, the set of vertices is a non-empty, at most countably infinite set, $\mu : G \times G \to \mathbb{R}$ are positive symmetric **weights** (also called conductances),

$$\mu_{x,y} = \mu_{y,x} \geq 0 \quad \forall x, y \in G$$

and $\kappa_x \geq 0$, $x \in G$ is a **killing measure**. The weights μinduce a set $\mathscr{E}$ of (undirected) edges, whereby $\{x, y\} \in \mathscr{E}$ if and only if (iff) $\mu_{x,y} > 0$. We write $x \sim y$ when $\{x, y\} \in \mathscr{E}$, and will always assume that T is **connected**, i,e. for all $x, y \in G$, there exists $n \geq 1$ and $x_i \in G$, $0 \leq i \leq n$ such that $x_0 = x$, $x_n = y$ and $x_{i-1} \sim x_i$ for all $1 \leq i \leq n$. Such a sequence $\gamma = (x_i)_{i-1}^n$ is called a **path** joining x and y (notation : $\gamma : x \leftrightarrow y$) and $\text{len}(\gamma) \equiv n$ is its length. We will also assume that T is **locally finite**, which means that $N(x) \equiv \{y \in G : y \sim x\}$ is a finite set for all $x \in G$.

We introduce the **graph (or chemical) distance** $d(\cdot, \cdot)$ on Γ, where $d(x, y) = \inf_{\gamma: x \leftrightarrow y} \text{len}(\gamma)$, for $x, y \in G$, and the balls

$$B(x, r) = \{y \in G : d(x, y) \leq r\}, \qquad x \in G, r \geq 0 .$$

Since Γ is connected, $d(\cdot, \cdot)$ is well-defined and it is a distance on G.

Given the data $\Gamma = (G, \mu, \kappa)$, we introduce the measure on G given by

$$\begin{cases} \mu(A) & = \sum_{x \in A} \mu_x , \qquad A \subseteq G \\ \mu_x & = \sum_{y \in G} \mu_{x,y} + \kappa_x , \qquad x \in G , (\text{the sum is finite}) \end{cases}$$

2.1 Random walk on Γ

The random walk on the weighted graph $\Gamma - (G, \mu, \kappa)$ is the Markov chain $X = (X_n)_{n \in \mathbb{N}}$ (with $\mathbb{N} = \{0, 1, 2, 3, \dots\}$) on $G \cup \{x_+\}$, where x_+ is a "cemetery" state (not in G) with transition probabilities

$$P(x, y) = \frac{\mu_{x,y}}{\mu_x} \quad x, y \in G , \qquad P(x, x_+) = \frac{\kappa_x}{\mu_x} , \quad x \in G ,$$

and $P(x_+, x_+) = 1$ (i.e. the state x_+ is absorbing).

Formally, by Kolmogorov's extension theorem there exists a **unique** probability measure P_x, for each $x \in G$, on the canonical space (Ω, a), where $\Omega = (G \cup \{x_+\})^{\mathbb{N}}$ and $a = \sigma(X_n, n \in \mathbb{N})$, where $X_n : \Omega \to G \cup \{x_+\}$ are the canonical coordinates

$$X_n(\omega) = \omega(n) , \qquad \text{for } \omega = (\omega(n))_{n \in \mathbb{N}} \in \Omega$$

such that, with $\mathscr{F}_n \equiv \sigma(X_0, X_0, \dots, X_n)$

$$\begin{cases} P_x(X_{n+1} = y | \mathscr{F}_n) & = P(X_n, y) , \qquad y \in G \cup \{x_+\} , n \geq 0 \\ P_x(X_0 = x) & = 1 . \end{cases} \tag{2.1}$$

We write E_x for the expectation corresponding to P_x, $E_x[f] = \int f \, \mathrm{d}P_x$.

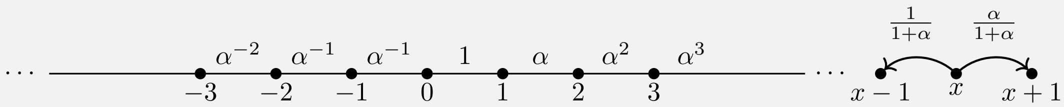
We now gives a list of examples of weighted graphs $\Gamma = (G, \mu, \kappa)$.

Examples 2.1. 1) $G = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $\alpha > 0$ fixed. Choose weights

$$\mu_{x,y} \equiv \alpha^{x \wedge y} , \text{ if } |x - y| = 1$$

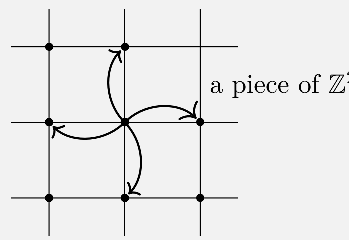
and $\mu_{x,y} = 0$ otherwise; $\kappa_x = 0$, for all $x \in G$. Here, $a \wedge b = \min\{a, b\}$ and $| \cdot |$ denotes the Euclidean distance. Note that $x \sim y$ iff $|x - y| = 1$ and $p_{x,x+1} = \frac{\alpha}{1+\alpha} = 1 - p_{x,x-1}$,

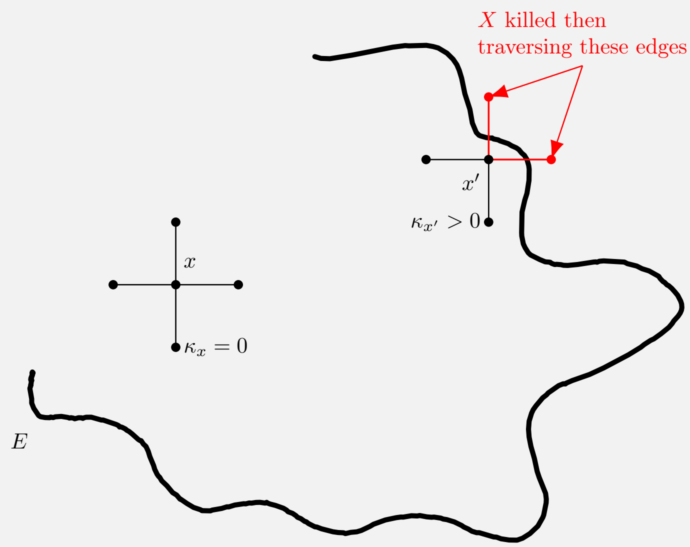
$\mu_x = \alpha^{x-1}(1+\alpha)\,, x \in \mathbb{Z}.$



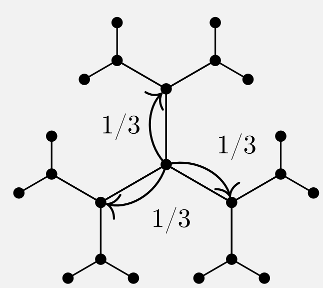
In particular, X is an asymmetric on $\mathbb{Z}$ when $\alpha \neq 1$ and symmetric (i.e. $X \overset{\text{law}}{=} -X$) when $\alpha = 1$.

2) Generally one has $G = \mathbb{Z}^d$, $d \in \{1, 2, 3, \dots\}$, $\mu_{x,y} = 1(|x-y| = 1)$ and $\kappa_{\cdot} = 0$ (so called **natural** weights). If $\kappa_x = \kappa > 0$, $x \in G$, then X is killed after N steps where N is a geometric random variable with parameter $\rho = \kappa/(2d+\kappa)$. One can also choose μ, κ to model RW killed outside a subset E of $G = \mathbb{Z}^d$: $\mu_{x,y} = 1 \quad \text{iff } |x-y| = 1\,, \quad x, y \in E$, $\kappa_x = \sum_{y \in \mathbb{Z}^d \setminus E} 1(|x-y| = 1)\,, \quad x \in G\,.$

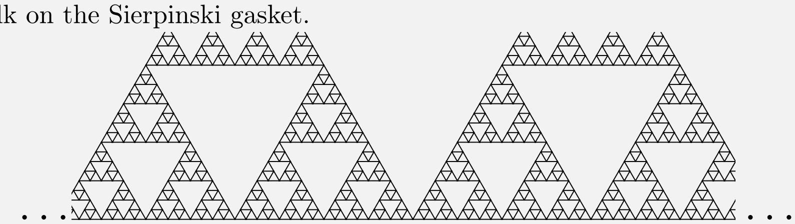




3) Random walk on the 3-regular tree $\mathbb{T}_3$ (with natural weights), and more generally on $\mathbb{T}_d$, the d-regular tree. When $d = 2$: see 1). When $d = 4$, $\mathbb{T}_4$ is the Cayley graph of $\mathbb{F}_2$ (free group of two elements).



4) Random walk on the Sierpinski gasket.



On Ω we have the **canonical shift** $\Theta_n : \Omega \to \Omega$, $n \geq 0$, defined by

$$\Theta_n(\omega)(\cdot) = \omega(n + \cdot)\,, \qquad n \geq 0\,,$$

in other words, the “trajectory shifted by n time units".

The **simple Markov property** (see [Nor98, Theorem 1.12], or [Dur19, p.284, Theorem 6.33] states that for Y a bounded random variable on (Ωa), for all $x \in G$, $n \geq 0$,

$$E_x[Y \circ \Theta_n | \mathscr{F}_n] = E_{X_n}[Y] \,, \qquad P_x - \text{a.s.} \tag{2.2}$$

(which boils down to (2.1) in the special case $Y = 1(x_1 = y)$). It can be reinforced into the **strong Markov property** (see [Dur19, p. 227, Theorem 6.5.4]), which states that for N an $(\mathscr{F}_n)$-stopping time, that is $\{N = n\} \in \mathscr{F}_n$ for each $n \geq 0$, one introduces the σ-algebra of the “ past of N",

$$\mathscr{F}_N = \{A \in a : A \cap \{N = n\} \in \mathscr{F}_n \quad \text{for all}\, n \geq 0\} \,.$$

The strong Markov property states that for Y as above, (2.2) holds and for N an $(\mathscr{F}_n)$-stopping time, all $x \in G$,

$$E_x[Y \circ \Theta_N | \mathscr{F}_N] \cdot 1(N < \infty) = E_{X_N}[Y] \cdot 1(N < \infty) \,, \qquad P_x - \text{a.s.} \tag{2.3}$$

(note $\{N < \infty\} \in \mathscr{F}_N$).

Typical examples of $(\mathscr{F}_n)$-stopping times are: for given $U \subseteq G$,

- the **entrance time in** U, $H_U \stackrel{\text{def}}{=} \inf\{n \geq 0 : X_n \in U\}$,
- the **exit time of** U, $T_U \stackrel{\text{def}}{=} \inf\{n \geq 0 : X_n \in G \setminus U\}$,
- the **hitting time of** U, $\tilde{H}_U \stackrel{\text{def}}{=} \inf\{n \geq 1 : X_n \in U\}$,

with the convention inf $= +\infty$. We abbreviate $H_x = H_{\{x\}}$ etc. when $U = \{x\}$, $x \in G$, is a singleton.

2.2 Green's functions

Fix a weighted graph $\Gamma = (G, \mu, \kappa)$, and for $x \in G$ let P_x be the law of the random walk (RW) $X = (X_n)_{n \in \mathbb{N}}$ on Γ and let $U \subseteq G$. We define the transition density of the RW **killed upon exiting** U as follows,

$$p_{n,U}(x,y) \stackrel{\text{def}}{=} \frac{P_x(X_n = y, n < T_U)}{\mu_y} \qquad n \geq 0, x, y \in G \,.$$

The factor $1/\mu_y$ is for convenience and in this instance one can view the above as the density of the killed RW with respect to μ. Note that $p_{n,U}(\cdot,\cdot)$ is symmetric in its arguments (by expressing the probability on the numerator as a sum over paths and using reversibility) and vanishes if either entry is in U.

We now come to a very important definition, namely, that of the Green's function associated to the weighted graph Γ.G

Definition 2.2 (Green's function)**.** *Let $U \subseteq G$. Then define the **Green's function** of U as*

$$g_U(x,y) \stackrel{\text{def}}{=} \sum_{n \geq 0} p_{n,U}(x,y) = \frac{1}{\mu_y} E_x \left[\sum_{n \geq 0} 1(X_n = y, n < T_U) \right] \,.$$

If $U = G$, write $g \equiv g_G$ and $p_n \equiv p_{n,U}$, $n \geq 0$.

The second equality shows that the Green's function of a subset U of G counts (up to rescaling by μ) the expected number of visits from x to y before exiting U.

Recall that a point $x \in G$ is called **recurrent** iff $P_x(X_n = x \text{ } i.o.) = 1$ and **transient** otherwise. This is a dichotomy in the sense if a point is transient then the above probability is zero. Moreover, recurrence and transience are class properties and so, in our case, by the assumed connectedness of Γ, we can say that Γ is either recurrent or transient if any point is either of the above.

Recurrence and transience are captured by the Green's function in the following way.

Proposition 2.3. *A weighted graph Γ is recurrent or transient if and only if there exist $x, y \in G$ such that $g(x,y) = \infty$ or $g(x,y) < \infty$, respectively.*

Proof. Exercise. $\square$

Remark. One is 'never too far from transience', however. This can be seen by showing that for all $x, y \in G$ and any **proper** subset $U \subsetneq G$, $g_U(x,y) < \infty$.

We now state and prove a lemma that gives the full Green's function in terms of the Green's function of the walk killed upon exiting U and residual terms.

Lemma 2.4. *One has for all $x, y \in G$, $U \subseteq G$*

$$g(x,y) = g_U(x,y) + E_x[1(T_U < \infty) \cdot g(X_{T_U}, y)] \,.$$

Proof. The cases where $U = G$ and the weighted graph Γ being recurrent are easily dealt with. Indeed, for the former $T_U = T_G = \infty$ almost surely and when Γ is recurrent, the Green's function $g(\cdot,\cdot)$ is always divergent.

Now, suppose Γ is transient and fix $U \subsetneq G$. By definition, we have for $x, y \in G$,

$$\begin{aligned}
g(x,y) &= g_U(x,y) + \frac{1}{\mu_y} E_x\left[\sum_{n \geq T_U} 1(X_n = y)\right] = g_U(x,y) + \sum_{n\geq 0} \frac{1}{\mu_y} E_x\left[\sum_{n \geq T_U} 1(X_n = y, T_U < \infty)\right] \\
&= g_U(x,y) + \sum_{n \geq 0} \frac{1}{\mu_y} E_x\left[1(T_U < \infty) \cdot \sum_{n\geq 0} 1(X_n = y) \circ \Theta_{T_U}\right] \\
&\overset{\text{SMP}}{=} g_U(x,y) + \frac{1}{\mu_y} E_{X_{T_U}}\left[1(T_U < \infty) \cdot \sum_{n \geq 0} 1(X_n = y)\right] \\
&= g_U(x,y) + g(X_{T_U}, y) \,.
\end{aligned}$$

$\square$

We now apply the above lemma in the case where U is the entire graph minus a singleton.

Examples 2.5. Let Γ be a transient weighted graph and let $U = G \setminus \{\zeta\}$, for $\zeta \in G$. Now, applying the above lemma with $x \in G$ arbitrary and $y = \zeta$, we obtain

$$g(x,y) = E_x[1(T_U < \infty) \cdot g(X_{T_U}, y)] = E_x[1(H_\zeta < \infty) \cdot g(\zeta, \zeta)]$$

as $X_{T_U} = \zeta$ and $T_U = H_{\{\zeta\}} \equiv H_\zeta$. Rearranging we obtain

$$P_x(H_\zeta < \infty) = \frac{g(x,\zeta)}{g(\zeta,\zeta)} \qquad x, \zeta \in G \,.$$

We now introduce the **equilibrium measure** associated to a subsets of the graph G.

Definition 2.6 (Equilibrium measure)**.** *Let $B \subset G$ be a finite set. Then for $y \in G$, define the measure on G,*

$$e_B(y) \stackrel{\text{def}}{=} P_y(\tilde{H}_B = \infty) \cdot 1_B(y)\,,$$

with total mass $\text{cap}(B) \equiv e_B(G)$, *which is called the* ***capacity*** *of* B.

We now prove the following result which the hitting probability of a set in terms of the Green's function and its equilibrium measure.

Theorem 2.7 (Last exit decomposition)**.** *Suppose Γ is transient and fix $B \subset G$ finite. Then, for $x \in G$, one has*

$$P_x(H_B < \infty) = Ge_B(x) \stackrel{\text{def}}{=} \sum_{y \in G} g(x,y) e_B(y)\,.$$

Proof. We will decompose the trajectory of the RW X on the time $n \geq 0$ (and location $y \in G$) of the last visit to B, $L_B = \sup\{n \geq 0 : X_n \in B\}$ ($< \infty$ almost surely, but is not a stopping time). We thus obtain

$$\begin{aligned}
P_x(H_B < \infty) &= P_x(H_B < \infty, L_B < \infty) \\
&= \sum_{n \geq 0, y \in B} P_x(X_n = y, X_k \notin B\ \forall k \geq n+1) \\
&= \sum_{n \geq 0, y \in B} E_x[1(X_n = y) \cdot 1(X_k \notin B\ \forall k \geq 1) \circ \Theta_n] \\
&\stackrel{\text{MP}}{=} \sum_{n \geq 0, y \in B} \frac{P_x(X_n = y)}{\mu_y} \cdot P_y(X_k \notin B\ \forall k \geq 1)\mu_y \\
&= \sum_{y \in G} g(x,y) \cdot e_B(y)\,.
\end{aligned}$$

□

Armed with our decomposition result, we now continue the previous example with the hitting times.

Examples 2.8. Claim: the capacitance of a singleton $\{\zeta\}$ for $\zeta \in G$, is given by

$$\text{cap}(\{\zeta\}) = e_{\{\zeta\}} = \frac{1}{g(\zeta,\zeta)}\,.$$

Indeed,

$$\begin{aligned}
e_{\{\zeta\}}\mu_\zeta &= P_\zeta(\tilde{H}_\zeta = \infty)\mu_\zeta \\
&\stackrel{\text{MP}}{=} \kappa_\zeta + \sum_{y \in G} \mu_{\zeta y} P_y(H_\zeta = \infty) \\
&= \kappa_\zeta + \sum_{y \in G} \mu_{\zeta y}(1 - P_y(H_\zeta < \infty)) \\
&\stackrel{\text{Thm}}{=} \mu_\zeta - \sum_{y \in G} \mu_{\zeta y} g(y,\zeta) \text{cap}(\{\zeta\})\,.
\end{aligned}$$

Rearranging now gives

$$\text{cap}(\{\zeta\}) = \mu_\zeta \left(1 + \sum_{y \in G} \mu_{\zeta y} g(y,\zeta)\right)^{-1} = g(\zeta,\zeta)^{-1}\,,$$

where the last equality follows from the application of Lemma 2.4 applied to $U = \{\zeta\}$.

We now prove the following result which gives a way of reconstructing the equilibrium measure of a finite subset of the ambient graph by a weighted average of hitting probabilities with respect to the equilibrium of a larger finite set containing it.

Theorem 2.9 (Sweeping/balayage identity)**.** *Fix $A \subseteq B \subset G$, both finite. Then,*

$$P_{e_B}(H_A < \infty, X_{H_A} = x) = e_A(x), \qquad x \in G\,.$$

Here, for a measure on G, we denote $P_\mu = \sum_{x\in G} \mu_x P_x$.

Proof. The main idea is again to obtain a path decomposition of the random walk and leverage the symmetry of the weights $\mu_{xy} = \mu_{yx}$, $x, y \in G$.

Now, observe that we can write for any $x \in G$, sectioning on the time $n \geq 0$ (and location $y \in G$) of the last exit from B (which is almost surely finite by the finiteness of B) and then using the Markov property to obtain,

$$e_A(x) = \sum_{n\geq 1, y\in G} \mu_x P_x(X_n \notin A\,, 1 \leq i \leq n, X_n = y) \cdot \frac{1}{\mu_y} e_B(y)\,.$$

For $n \geq 1$, consider the collection of paths

$$\gamma_n \overset{\text{def}}{=} \{(x_i)_{i=0}^n\,, x_i \in G\,, 1 \leq i \leq n : x_0 = x\,, x_n = y\,, x_i \sim x_{i-1}\,, x_i \notin A\,, 1 \leq i \leq n\}\,.$$

We can now re-express (by by summing over paths and using the Markov property and a quick induction on the path length) the above as

$$\begin{aligned} e_A(x) &= \sum_{n\geq 1, \gamma_n} \mu_{x_0} \left(\prod_{i=1}^n \frac{\mu_{x_{i-1}x_i}}{\mu_{x_{i-1}}} \right) \cdot \frac{1}{\mu_{x_n}} e_B(x_n) \\ &= \sum_{n\geq 1, \gamma_n} \left(\prod_{i=1}^n \frac{\mu_{x_{i-1}x_i}}{\mu_{x_i}} \right) e_B(x_n) \\ &= \sum_{x\in G} P_{e_B}(H_A < \infty, X_{H_A} = x) e_B(x)\,, \end{aligned}$$

by reversibility, concluding the proof. □

2.3 Laplacian and the Dirichlet form

We now establish some notation regarding certain function spaces that we will encounter in this section. We begin with a definition.

Definition 2.10. *Let $\Gamma = (G, \mu, \kappa)$ be a weighted graph. Call the* ***space of functions***

$$\mathscr{C}(G) = \mathbb{R}^G = \{f : G \to \mathbb{R}\} ,$$

the ***space of compactly supported functions***

$$\mathscr{C}_c(G) = \mathbb{R}^G = \{f \in \mathscr{C}(G) : \{x \in G : f(x) \neq 0\} \text{, is finite}\} ,$$

and the space of ***function with finite p-moment****,*

$$L^p(\mu) = \{f \in \mathscr{C}(G) : \|f\|_p^p \equiv \sum_{x \in G} |f_x|^p \mu_x < \infty\} .$$

For the random walk X on Γ, we will call the **transition operator** of X $P : \mathscr{C}(G) \to \mathscr{C}(G)$, given by

$$Pf(x) = E_x[f(X_1)] = \sum_{y \in G} P(x,y) f(y) = \frac{1}{\mu_y} \sum_{y \in G} \mu_{x,y} f(y) , \qquad \text{for all } x \in G, f \in \mathscr{C}(G) ,$$

which is well-defined since the graph Γ is locally finite, meaning for all $x \in G$, the sum above is finite.

We now define the Laplacian of a function $f \in \mathscr{C}(G)$.

Definition 2.11. *Let $f \in \mathscr{C}(G)$. The* ***Laplacian*** *is the operator $\Delta : \mathscr{C}(G) \to \mathscr{C}(G)$*

$$\Delta f = (P - I) f , \qquad \text{for all } f \in \mathscr{C}(G) ,$$

where I is the identity map.

Remark. For $f \in \mathscr{C}(G)$, we can write using the expression for the transition operator P,

$$\Delta f(x) = \frac{1}{\mu_x} \sum_{y \in G} \left(\mu_{x,y}(f(y) - f(x)) - \kappa_x f(x) \right) .$$

This implies that one can think of the Laplacian as a 'local averaging operator' (ignoring the killing).

Now, we prove the following lemma, which states that one can construct a 'right-inverse' of the Laplacian using the Green's function.

Lemma 2.12. *Suppose Γ is a transient weighted graph and let $f \in L^1(\mu)$. Then one has the identity as elements of $\mathscr{C}(G)$,*

$$-\Delta G f = f .$$

Remark. With $f(\cdot) = 1/\mu_y 1_y$, one easily computes $Gf(\cdot) = g(\cdot, y) \equiv g^y(\cdot)$ for all $y \in G$. Hence, the above lemma implies that

$$-\Delta g^y = \frac{1}{\mu_y} 1_y , \qquad y \in G.$$

The Green's function is what is called the **fundamental solution** of the heat equation on G.

Proof. Since Γ is transient, we have for all $x \in G$, $g(x,x) < \infty$, and Lemma 2.4 $g(y,x) \leq g(x,x)$, for all $x, y \in G$. This gives

$$\sum_{y \in G} g(x,y) |f(y)| \mu_y \leq g(x,x) \cdot \|f\|_1 < \infty ,$$

and in particular that all sums considered below will be absolutely convergent. Now, we compute

$$\begin{aligned} Gf(x) = \sum_{y\in G} g(y,x)f(y)\mu_y &= \sum_{n\geq 0}\sum_{y\in G} P_x(X_n = y)f(y) = \sum_{n\geq 0} E_x[f(X_n)] \\ &= f(x) + \sum_{n\geq 0} E_x[f(X_n)\circ\Theta_1] \\ &\overset{\text{MP}}{=} f(x) + \sum_{y\in G} P(x,y)\sum_{n\geq 0} E_y[f(X_n)] \\ &\overset{\text{MP}}{=} f(x) + \sum_{y\in G} P(x,y)Gf(y)\,. \end{aligned}$$

To finish the proof, just rearrange the above equality and use the identity $-\Delta = I - P$. □

We now define an important quantity, the **Dirichlet form** associated to Γ, which quantifies how 'variable' functions over G are.

Definition 2.13. *Let $f, g \in \mathscr{C}(G)$ be such that*

$$\sum_{x,y\in G} |f(y) - f(x)|\cdot|g(y) - g(x)|\mu_{x,y} + \sum_{x\in G}\kappa_x|f(x)|\cdot|g(x)| < \infty\,. \tag{C1}$$

Define the ***Dirichlet form*** *acting on f, g as*

$$\mathscr{E}(f,g) \overset{\text{def}}{=} \frac{1}{2}\sum_{x,y\in G}(f(y) - f(x))\cdot(g(y) - g(x))\mu_{x,y} + \sum_{x\in G}\kappa_x f(x)\cdot g(x)$$

We will use the notation $(f,g) = \sum_{x\in G} f(x)g(x)\mu_x$, whenever the sum is absolutely convergent.

We now state and prove an integration-by-parts formula functions on G.

Proposition 2.14 (Gauss-Green, integration-by-parts)**.** *Let $f, g \in \mathscr{C}(G)$ be such that*

$$\sum_{x,y\in G} |f(x)|\cdot|g(y) - g(x)|\mu_{x,y} + \sum_{x\in G}\kappa_x|f(x)|\cdot|g(x)| < \infty\,. \tag{C2}$$

Then, we have (2.13) *is satisfied and*

$$\mathscr{E}(f,g) = (f, -\Delta g)\,.$$

Proof. By (C2), all terms below will be absolutely convergent, so well-defined. We now compute,

$$\begin{aligned} (f,\Delta g) &= \sum_{x,y\in G} f(x)(g(y) - g(x))\mu_{x,y} - \sum_{x\in G}\kappa_x f(x)g(x) \\ &= \frac{1}{2}\sum_{x,y\in G} f(x)(g(y) - g(x))\mu_{x,y} + \frac{1}{2}\sum_{x,y\in G} f(y)(g(x) - g(y))\mu_{y,x} - \sum_{x\in G}\kappa_x f(x)g(x) \quad (\mu_{\cdot,\cdot'} = \mu_{\cdot',\cdot}) \\ &= \frac{1}{2}\sum_{x,y\in G}(f(x) - f(y))(g(y) - g(x))\mu_{x,y} - \sum_{x\in G}\kappa_x f(x)g(x) \\ &= -\mathscr{E}(f,g)\,. \end{aligned}$$

□

We now introduce the space of all functions with finite Dirichlet form (energy) and study some of its properties. Formally, we consider the set

$$H = H(G, \mu, \kappa) = \{f \in \mathscr{C}(G) : \mathscr{E}(f, f) < \infty\} .$$

By Cauchy-Schwarz, for any $f, g \in H$, one can check that $\mathscr{E}(f, g)$ is well-defined, giving that H is a linear space under pointwise addition and scalar multiplication.

Now, pick any $x^* \in G$, and define the inner-product

$$(f, g)_H \equiv \mathscr{E}(f, g) + f(x^*)g(x^*) , \qquad \text{for all } f, g \in H ,$$

and norm

$$\|f\|_H^2 \equiv (f, f)_H , \qquad \text{for all} f \in H .$$

The second term in the definition of the inner product is necessary for $\|\cdot\|_H \equiv (\cdot,\cdot)_H$ to be a proper norm and not a semi-norm (this term is not necessary when the killing field $\kappa_{\cdot} \not\equiv 0$).

We now prove some structural properties of the space H.

Proposition 2.15. *(i): For all $f \in L^2(\mu)$, $\mathscr{E}(f, f) \leq 2\|f\|_2^2 (< \infty)$, and so one has the inclusion $L^2(\mu) \subseteq H$;*

(ii): for all $x \in G$, here exists some $0 < c(x) < \infty$ such that $|f(x)| \leq c(x) \cdot \|f\|_H$;

(iii): $(H, (\cdot,\cdot)_H)$ is a (separable) Hilbert space.

Proof. (i) follows by Cauchy-Schwarz.

To prove (ii), observe first that if $x = x^*$, then taking $c(x) = c(x^*) = 1$ works. Now, if $x \neq x^*$, by the connectedness of Γ, one can pick the existence of a path γ of length $n \geq 1$ from x to x_*,

$$\gamma = (x_i)_{i=0}^n , x_i \in G , 1 \leq i \leq n : x_0 = x , x_n = y , x_i \sim x_{i-1} , 1 \leq i \leq n .$$

Then, one estimates

$$\begin{aligned} |f(x)| &\leq |f(x^*)| + \sum_{i=1}^n |f(x_i) - f(x_{i-1})| && \text{(triangle inequality)} \\ &\leq |f(x_*)| + \left(\sum_{i=1}^n \frac{1}{\mu_{x_i,x_{i-1}}}\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^2 \mu_{x_i,x_{i-1}}\right)^{\frac{1}{2}} . && \text{(Cauch-Schwarz)} \end{aligned}$$

Now, one can take $c(x) = 1 + (\sum_{i=1}^n 1/\mu_{x_i,x_{i-1}})^{\frac{1}{2}}$.

To prove (iii), it is clear that the space $(H, (\cdot,\cdot)_H)$ is an inner-product space, as remarked with the use of Cauchy-Schwarz. For completeness, one leverages the pointwise bounds in (ii) and Fatou's lemma to obtain L^2 convergence to the candidate limit. Finally, separability follows from the discreteness of the underlying space G. □

Remark. 1): The inclusion $L^2(\mu) \subset H$ in general is strict. For instance, one can take $G = \mathbb{Z}^d$, $d \geq 1$ with natural weights (only need $\mu(G) = \infty$); then $1 \notin L^2(\mu)$, but $1 \in H$, as $\mathscr{E}(1, 1) = 0$;

2): (C2) is satisfied if either f or g are in $\mathscr{C}_c(G)$, and also if $g \in H$ and $f \in L^2$ (as $\Delta g \in L^2$);

3): Proposition 2.14 does **NOT** hold for all $f, g \in H$. For instance, let $G = \mathbb{Z}$, μ =natural weights, $f = 1 \in H$, and

$$g(x) = 1(x \geq 1) \cdot \sum_{1 \leq k \leq x} \frac{(-1)^k}{k} .$$

Notice that $g \in H$, yet

$$|\Delta g(x)| = \left|\frac{g(x+1) + g(x-1)}{2} - g(x)\right| \sim \frac{1}{x} , x \to \infty ,$$

and so

$$\sum_{x\in\mathbb{Z}} |\Delta g(x)| \cdot \underbrace{|f(x)|}_{\equiv 1} \cdot \underbrace{\mu_x}_{\equiv 2} = +\infty$$

and cannot be meaningfully defined.

2.4 variational principles

Let H_0 denote the **homogeneous** version of the space H, defined as

$$H_0 \overset{\text{def}}{=} \text{ the } \|\cdot\|_H - \text{closure of } \mathscr{C}_c(G)(\text{or equivalently } L^2(G)) \subseteq H\,.$$

We now prove a useful proposition.

Proposition 2.16. *Let Γ be a transient weighted graph. Then,*

- *for all $x_0 \in G$, $g^{x_0}(\cdot) \in H_0$,* $(\star)$
- *for all $x_0 \in G$, $f \in H_0$, $\mathscr{E}(g^{x_0}, f) = f(x_0)$.*

Proof. Recall the notation for $n \geq 1$, $B_n = B(x_0, n)$ for the balls in G with respect to the graph distance. Observe that

$$g^{x_0}_{B_n} \text{ is supported on } B_n \text{ and so is in } \mathscr{C}_c(G)\,, \text{ for all } n \geq 1\,.$$

Furthermore, for all $1 \leq \ell \leq n$, by integration by parts (justified as all sums are finite)

$$\mathscr{E}(g^{x_0}_{B_\ell}, g^{x_0}_{B_n}) = (g^{x_0}_{B_\ell}, \underbrace{-\Delta g^{x_0}_{B_n}}_{\mu_{x_0}^{-1} 1_{x_0}}) = g^{x_0}_{B_\ell}(x_0) = g_{B_\ell}(x_0, x_0)\,.$$

Now, by the bi-linearity of the Dirichlet form, we have

$$\begin{aligned}\mathscr{E}(g^{x_0}_{B_n} - g^{x_0}_{B_\ell}, g^{x_0}_{B_n} - g^{x_0}_{B_\ell}) &= \mathscr{E}(g^{x_0}_{B_\ell}, g^{x_0}_{B_\ell}) - 2\mathscr{E}(g^{x_0}_{B_\ell}, g^{x_0}_{B_n}) + \mathscr{E}(g^{x_0}_{B_n}, g^{x_0}_{B_n}) \\ &= g_{B_n}(x_0, x_0) - g_{B_\ell}(x_0, x_0) \to 0\,, n, \ell \to \infty\,. \qquad \text{(transience of } \Gamma)\end{aligned}$$

It now follows that $(g^{x_0}_{B_n})_{n\geq 1}$ is Cauchy in H, and so by completeness, $g^{x_0}_{B_n} \to \overline{g}$ in H and by Proposition 2.15 (pointwise convergence), $\overline{g} = g^{x_0}$.

Now, for $f \in H_0$, let $\mathscr{C}_c(G) \ni f_n \overset{\|\cdot\|}{\to} f$ be an approximating sequence, then by Cauchy-Schwarz $\mathscr{E}(g^{x_0}, f_n) \to \mathscr{E}(g^{x_0}, f)$ as $n \to \infty$, but by integration by parts, $\mathscr{E}(g^{x_0}, f_n) = f_n(x_0) \to f(x_0)$ as $n \to \infty$. □

Remark. 1) Consider $\mathbb{Z}^3$ with natural weights; we know that $g^0(x) \asymp 1/|x| \vee 1$, $x \in \mathbb{Z}^3$. Fro this one sees that $g^0 \notin \mathbb{Z}^3$, but $g^0 \in H$, so the inclusion $H \subseteq L^2$ is strict.

2) The inclusion $H_0 \subseteq H$ can also turn out to be strict. For instance, if we take $f \equiv 1$ on a transient weighted graph Γ, we see that if $f \in H_0$,

$$1 = f(x_*) + \mathscr{E}(f,f) = \mathscr{E}(g^{x_*}, f) + \mathscr{E}(f,f) \overset{\text{C}-\text{S}}{\leq} \mathscr{E}(f,f)(1 + \underbrace{\mathscr{E}(g^{x_*}, g^{x_*})}_{=g(x_*, g_{x_*})}) = 0\,,$$

a contradiction.

We now prove a variational characterisation of the capacity of subsets of G involving the Dirichlet energy of a subclass of functions in H_0.

Theorem 2.17. *Let $A \subseteq G$ be finite. Then, we have the variational characterisation*

$$\text{cap}(A) = \inf\{\mathscr{E}(f,f) : f \equiv 1 \text{ on } A\,, f \in H_0\}\,.$$

Proof. Consider the function $V(x) = P_x(H_A < \infty)$, $x \in G$. One can argue like for $(\star)$ that $V \in H_0$.

If Γ is recurrent, one has $V \equiv 1$ and so $\mathscr{E}(V,V) = 0$, which gives that $\text{cap}(A) = 0$, by the non-negativity of the Dirichlet energy.

Now, suppose Γ is transient. Then we compute

$$\begin{aligned}
\mathscr{E}(V,V) &= \mathscr{E}(V, Ge_A)\,, && \text{(last exit decomposition)}\\
&= \sum_{x\in G} e_A(x)\mathscr{E}(V,g^x)\,, && (A \text{ finite})\\
&= \sum_{x\in G} e_A(x) \underbrace{V(x)}_{=1, x\in A} = \text{cap}(A)\,. && \text{(Proposition 2.16)}
\end{aligned}$$

Now, fr $f \in H_0$, $f \equiv 1$ on A, we have

$$\mathscr{E}(f-V,V) = \sum_{x\in G} e_A(x)\cdot(f-V)(x) = 0\,. \qquad (e_A \text{ supported on } A)$$

Finally, by bi-linearity

$$\mathscr{E}(f,f) = \underbrace{\mathscr{E}(f-V,f-V)}_{\geq 0} + 2\underbrace{\mathscr{E}(f-V,V)}_{=0} + \mathscr{E}(V,V) \geq \mathscr{E}(V,V)\,.$$

□

For ν, ν' finitely supported measures on G, define

$$E(\nu,\nu') \stackrel{\text{def}}{=} \sum x, y \in G\nu(x)g(x,y)\nu'(y)\,.$$

We now obtain a 'dual' version of the above for the inverse capacity formulated with measures.

Theorem 2.18. *Let $A \subseteq G$ be finite. Then, we have the variational characterisation*

$$\frac{1}{\text{cap}(A)} = \inf\{E(\nu,\nu) : \nu \text{ probability measure supported on } A\}\,.$$

Proof. In the case Γ is recurrent, there is nothing to show as the Green's function is always divergent.

Suppose Γ is transient. Define the normalised equilibrium measure $\overline{e}_A(\cdot) = e_A(\cdot)/\text{cap}(A)$. Then, we compute, for all probability measures ν supported on A,

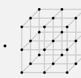
$$\begin{aligned}
E(\nu,\overline{e}_A) &= \sum_{x,y\in G} \nu(x)g(x,y)\frac{e_A(y)}{\text{cap}(A)}\\
&= \sum_{x\in G} \nu(x)\frac{\overbrace{P_x(H_A<\infty)}^{=1}}{\text{cap}(A)} = \frac{1}{\text{cap}(A)}\,. && \text{(last exit decomposition)}
\end{aligned}$$

Finally, we have by bi-linearity,

$$E(\nu,\nu) = \underbrace{E(\nu-\overline{e}_A,\nu-\overline{e}_A)}_{\geq 0(!!)} + 2\underbrace{E(\nu-\overline{e}_A,\overline{e}_A)}_{=0} + \underbrace{E(\overline{e}_A,\overline{e}_A)}_{=\text{cap}(A)^{-1}} \geq \frac{1}{\text{cap}(A)}\,.$$

□

Examples 2.19. Consider again $\mathbb{Z}^3$ with natural weights. For $n \geq 1$, let $B_n = [-n, n]^d \cap \mathbb{Z}^d$.



We know that for $d \geq 3$, the Green's function has the following asymptotics,

$$c(|x-y|^{2-d} \wedge 1)) \leq g(x, y) \leq c'(|x-y|^{2-d} \wedge 1\,,$$

for all $x, y \in \mathbb{Z}^d$ and some constants $c, c' > 0$.

Claim: there exist constants $\tilde{c}, \tilde{c}'$ such that

$$\tilde{c} n^{d-2} \leq \operatorname{cap}(A) \leq \tilde{c}' n^{d-2}\,.$$

Indeed, we first establish the lower bound using Theorem 2.18 with probability measure $\nu = 1/|B_n| 1_{B_n}$. We can compute

$$\begin{aligned} E(\nu, \nu) = \frac{1}{|B_n|^2} \sum_{x,y \in B_n} g(x, y) &\leq \frac{1}{|B_n|} \sum_{y \in B_n} \sup_{x \in B_n} g(x, y) \\ &\leq c n^{-d} \sum_{k=1}^{3n} k^{d-1} \cdot k^{2-d} \leq c n^{2-d}\,, \end{aligned}$$

for some constant $c > 0$. Observe that here, ν being uniform inside B_n is nothing like the equilibrium measure, which would only be supported on the boundary of the box B_n.

For the upper bound, the naive guess $f_n = 1_{B_n}$ gives $\mathscr{E}(f, f) = \mathscr{O}(n^{d-1})$. This is off the exponent of the lower bound by one. When choosing a good proxy f to use to compute the capacitance of a set, one needs to strike a balance between the variability of the gradient and the volume of the support of the function. Let us try to do this below, choosing, for $n \geq 1$,

$$f_n(x) = 1_{B_n}(x) + \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) 1_{B_{n+k} \setminus B_{n+k-1}}(x)\,, \qquad x \in \mathbb{Z}^d\,.$$

Essentially, f_n decays linearly on a family of increasing shells surrounding B_n. One can estimate the Dirichlet energy to obtain,

$$\begin{aligned} \mathscr{E}(f_n, f_n) &\leq cd \sum_{k=0}^{n-1} |B_{n+k} \setminus B_{n+k-1}| \left(\left(1 - \frac{k+1}{n}\right) - \left(1 - \frac{k}{n}\right) \right)^2 \\ &\leq c' |B_{2n} \setminus B_{2n-1}| \sum_{k=0}^{n-1} \frac{1}{n^2} \leq c'' n^{d-1} \cdot n \cdot n^{-2} \leq c'' n^{d-2}\,. \end{aligned}$$

3 The Gaussian Free Field

Consider a transient weighted graph $\Gamma = (G, \mu, \kappa)$. The goal will be to construct a probability measure on the space of functions $\mathbb{R}^G$ equipped with the cylindrical sigma algebra $\sigma(\varphi_x : x \in G)$, where for $x \in G$, φ_x is the evaluation map $\mathbb{R}^G \ni f \mapsto \varphi_x(f) = f(x)$, such that for all $x \in G$,

$$\begin{cases} (\varphi_x)_{x \in G} & \text{is a } \textbf{Gaussian field} \\ \mathbb{E}[\varphi_x \varphi_y] = g(x, y)\,, & \text{for all } x, y \in G\,, \end{cases} \tag{GFF}$$

where $g(\cdot,\cdot)$ denotes, as usual, the Green's function associated to Γ, which, by transitivity is finite for all entries. By **Gaussian field** we mean that when we 'test' φ against a finitely supported probability measure ν on G, the expression $\langle \nu, \varphi \rangle = \sum_{x\in G} \nu_x \varphi_x$ is a mean zero Gaussian with variance

$$\mathbb{E}[\langle \nu, \varphi \rangle^2] = \sum_{x,y\in G} \nu_x \nu_y g(x,y) = E(\nu,\nu)\,.$$

This leads us to the following definition.

Definition 3.1. *Let $\varphi = (\varphi_x)_{x\in G}$ be as above. Then, φ is called the **Gaussian Free Field**(of Γ).*

Remark. In some sense, the Gaussian free field of a transient weighted graph can be seen as 'the canonical random surface'.

In the following theorem, we will show that one can indeed construct such the Gaussian free field.

Theorem 3.2. *Let $\Gamma = (G, \mu, \kappa)$ be a transient weighted graph. Then, there exists a unique law $\mathbb{P}$ on $(\mathbb{R}^G, \sigma(\varphi_x : x \in G))$ satisfying* (GFF).

Proof. Uniqueness is easy. For any finite $A \subset G$, observe that (GFF) uniquely specifies the characteristic functions of $\varphi|_A$, a finite-dimensional Gaussian vector, since for all ν measures supported on A, we have $\mathbb{E}[\mathrm{e}^{i\langle \nu,\varphi\rangle}] = \mathrm{e}^{-1/2E(\nu,\nu)}$. Since the collection of preimages of Borel sets under the evaluation maps $\varphi|_U$, $U \subset G$ finite generate the cylindrical sigma algebra above, one has by standard measure-theoretic arguments that the law $\mathbb{P}$, if it exists, is uniquely specified.

Existence is more subtle. Even when proving uniqueness, we have have ignored the fact that we need to show the bilinear from $E(\cdot,\cdot)$ is positive definite. This will follow from our proof of existence. The strategy will be to embed G in a larger graph $\tilde{G}$ in a way that will allow us to 'take the square root' of the Green's function on G by expressing it in terms that of $\tilde{G}$ and then define the GFF explicitly.

Now, consider the 'augmented graph' $\tilde{\Gamma} = (\tilde{\Gamma}, \tilde{\mu}, \tilde{\kappa})$, which we define as follows: for each $x, y \in \mathscr{E}$, introduce the vertex $\zeta_{x,y}$ and weights

$$\tilde{\mu}_{x,\zeta_{x,y}} = \tilde{\mu}_{\zeta_{x,y},x} = \tilde{\mu}_{\zeta_{x,y},y} = \tilde{\mu}_{y,\zeta_{x,y}} = 2\mu_{x,y}\,.$$

Moreover, if $\kappa_x > 0$, we set $\tilde{\kappa_x} = 0$ and introduce a new vertex ζ_x and weights

$$\tilde{\mu}_{x,\zeta_x} = \tilde{\mu}_{\zeta_x,x} = 2\kappa_x\,, \tilde{\kappa}_x = 2\kappa_x\,.$$

Finally, the vertex set is given by

$$\tilde{G} = \left\{ \zeta : \zeta = \begin{matrix} \zeta_{x,y}\,, & x,y\in G\,, \mu_{x,y} > 0 \\ \zeta_x\,, & x \in G\,, \kappa_x > 0 \end{matrix} \right\}\,.$$

Let $\tilde{X}$ and $\tilde{P}_x$, $x \in \tilde{G}$, $\tilde{g}(\cdot,\cdot)$ denote the random walk on $\tilde{\Gamma}$, its associated canonical laws and its Green's function, respectively.

We prove a preliminary lemma.

Lemma 3.3. *$g(x,y) = \tilde{g}(x,y)$, for all $x, y \in G$.*

Proof. Consider $x \in G$, then we compute the two step transition probabilities of $\tilde{X}$ as,

$$P_x(\tilde{X}_2 = x) = \sum_{\zeta \in \tilde{G}} \frac{\tilde{\mu}_{x,\zeta}}{\tilde{\mu}_x} \cdot \frac{\tilde{\mu}_{\zeta,x}}{\tilde{\mu}_\zeta} = \sum_{y \in G} \frac{(2\mu_{x,y})^2}{2\mu_x 4\mu_{x,y}} + \frac{(2\kappa_x)^2}{2\mu_x 4\mu_x} = \frac{1}{2}\,,$$

$$P_x(\tilde{X}_2 = y) = \frac{\tilde{\mu}_{x,\zeta_{x,y}}}{\tilde{\mu}_x} \cdot \frac{\tilde{\mu}_{\zeta_{x,y},x}}{\tilde{\mu}_x} = \frac{(2\mu_{x,y})^2}{2\mu_x 4\mu_{x,y}} = \frac{1}{2} P(x,y)\,, G \ni y \sim x\,.$$

The above dynamics lend us a picture of $\tilde{X}$ being a 'lazy' version of X, having a propensity to stay put at every step with probability $1/2$. Continuing in this fashion, we obtain inductively that for all $n \geq 0$,

$$\tilde{X}_{2n} \overset{\text{Law}}{=} X_{N_n}\,,$$

where $N_n \sim \mathrm{Bin}(n, 1/2)$ is independent of X. This makes sense by also considering the fact that at even times $\tilde{X}$, starting in G is always constrained to stay in G (or go to the same graveyard state as X). Hence, for $x, y \in G$, another computation gives

$$\tilde{g}(x,y)\tilde{\mu}_y = \sum_{n \geq 0} \tilde{P}_x(X_{2n} = y) = \sum_{n \geq 0} P_x(X_{N_n} = y) = \sum_{k \geq 0} \underbrace{\left(\sum_{n \geq k} \binom{n}{k} \cdot \frac{1}{2^n} \right)}_{= \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{x^k}{1-x} \big|_{x=\frac{1}{2}}}$$

$$= 2 \sum_{k \geq 0} P_x(X_k = y) = 2g(x,y)\mu_y = g(x,y)\tilde{\mu}_y\,.$$

□

Now, on a suitable probability space, one can construct the family of iid. $\mathscr{N}(0,1)$ (centred variance one Gaussians) $(Y_{n,x})_{n \geq 0, x \in \tilde{G}}$ and define for $k \geq 1$,

$$\varphi_x^k \overset{\text{def}}{=} \sum_{0 \leq n \leq k} \sum_{y \in \tilde{G}} \frac{1}{\sqrt{\tilde{\mu}_y}} \tilde{P}_x(\tilde{X}_n = y) \cdot Y_{n,y}\,, \qquad x \in G\,.$$

Notice that for al $k \geq 0$, the above expression is a well-defined centred Gaussian since that law $\tilde{P}_x(X_n = \cdot)$, $n \geq 0$ is finitely supported (since we assume our graphs are locally finite) and for any finitely supported measure ν, we compute

$$\begin{aligned}
\mathbb{E}[|\langle \nu, \varphi^k \rangle|^2] &= \sum_{x,x' \in G} \nu_x \nu_{x'} \sum_{n,n'=0}^{k} \sum_{y,y' \in \tilde{G}} \frac{1}{\sqrt{\tilde{\mu}_y \tilde{\mu}_{y'}}} \tilde{P}_x(\tilde{X}_n = y) \tilde{P}_{x'}(\tilde{X}_{n'} = y') \cdot \underbrace{\mathbb{E}[Y_{n,y} Y_{n',y'}]}_{= \delta_{n,n'} \cdot \delta_{y,y'}} \\
&= \sum_{x,x' \in G} \nu_x \nu_{x'} \sum_{n=0}^{k} \sum_{y \in \tilde{G}} \frac{1}{\tilde{\mu}_y} \tilde{P}_x(\tilde{X}_n = y) \tilde{P}_{x'}(\tilde{X}_n = y) \\
&= \sum_{x,x' \in G} \nu_x \nu_{x'} \sum_{n=0}^{k} \sum_{y \in \tilde{G}} \tilde{p}_n(x,y) \tilde{p}_n(y,x') && \text{(Markov property)} \\
&= \sum_{x,x' \in G} \nu_x \nu_{x'} \sum_{n=0}^{k} \tilde{p}_{2n}(x,x') \\
&\overset{k \to \infty}{\nearrow} \sum_{x,x' \in G} \nu_x \nu_{x'} \sum_{n=0}^{\infty} \tilde{p}_{2n}(x,x') = E(\nu,\nu)\,. && \text{(Lemma 3.3)}
\end{aligned}$$

This now establishes the positive definiteness of the bilinear form $E(\cdot,\cdot)$, and by Kolmogorov's two series theorem, allows us to **define** the Gaussian Free Field φ as

$$\varphi_x \stackrel{\text{def}}{=} \sum_{n\geq 0}\sum_{y\in\tilde{G}} \frac{1}{\sqrt{\tilde{\mu}_y}} \tilde{P}_x(\tilde{X}_n = y)\cdot Y_{n,y}\,, \qquad x\in G\,. \tag{3.1}$$

□

Remark. The above construction of the Gaussian free field in terms of iid. centred unit variance Gaussians is known as the 'white noise' expansion of the GFF, and is often useful in practice.

The proof of the above theorem also gives for any finitely supported $\nu :\to \mathbb{R}$,

$$\mathbb{E}(\nu,\nu) = \sum_{x,x'\in G} \nu_x\nu_{x'} \sum_{n=0}^{\infty} \tilde{p}_{2n}(x,x') = \sum_{n=0}^{\infty}\sum_{y\in\tilde{G}} \left(\sum_{x\in G}\tilde{p}_n(x,y)\sqrt{\tilde{\mu}_y}\right)\cdot\left(\sum_{x'\in G}\tilde{p}_n(x',y)\sqrt{\tilde{\mu}_y}\right) \geq 0\,.$$

We will now investigate the Gaussian free field on subsets of general connected, weighted graphs $\Gamma = (G,\mu,\kappa)$, which will give us insight into the local dynamics and look at some simple but important examples, revealing familiar objects.

Now, consider some $U \subsetneq G$, then we know the Green's function $g_U(x,y) < \infty$, for all $x,y\in G$. Write $\mathbb{P}_U$ to be the law of the GFF such that it is, again, a centred Gaussian field on G with

$$\mathbb{E}[\varphi_x\varphi_y] = g_U(x,y)\,, \qquad \text{for all } x,y\in G\,.$$

The law $\mathbb{P}_U$ is already accessible to use from Theorem 3.2. Indeed, observe that one can set $\varphi|_{G\setminus U\equiv 0}$ $\mathbb{P}_U$-almost surely, and let $(\varphi_x)_{x\in U}$ be the GFF associated to the transient, connected $\Gamma_U \equiv (U, \mu|_{U\times U}, \kappa^U)$, where κ^U is the killing measure associated to exiting the domain U in G (see illustration 2.1, with $E\equiv U$).

It is instructive to see what happens when $U\subseteq G$ is finite, and with $g_U(\cdot,\cdot)<\infty$. Clearly, from the white-noise decomposition (3.1) of the GFF, we have for finite U, that $(\varphi_x)_{x\in U}$ is a Gaussian vector in $\mathbb{R}^U$. Do we have an explicit form for its law? The answer is yes, and to see this, first recall the definition for the Dirichlet form associated to U, for $f\in\mathbb{R}^U$,

$$\mathscr{E}_U(f,f) = \frac{1}{2}\sum_{x,y\in U}\mu_{x,y}(f(y)-f(x))^2 + \sum_{x\in U}\kappa_x + \left(\sum_{y\in G\setminus U}\mu_{x,y}\right)(f(x))^2\,.$$

We prove in the following proposition that $(\varphi_x)_{x\in U}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^U$, which we can write down explicitly.

Proposition 3.4. *Let $U\subseteq G$ be finite with $g_U(\cdot,\cdot)<\infty$. Then, $(\varphi_x)_{x\in U}$ under $\mathbb{P}_U$ has law*

$$\frac{e^{-\frac{1}{2}\mathscr{E}_U(\varphi,\varphi)}}{(2\pi)^{\frac{|U|}{2}}(\det(G_U))^{\frac{1}{2}}} \quad\cdot\quad \underbrace{\prod_{x\in U} d\varphi_x}_{\textit{Leb. meas. on } \mathbb{R}^U}\,,$$

where G_U is the $|U|\times|U|$ matrix with entries $G_U = (g_U(x,y))_{x,y\in U}$.

Remark. 1) Under the above assumption, one always has

$$\kappa_\cdot^U = \kappa_\cdot + \sum_{y\in G\setminus U}\mu_{\cdot,y} \not\equiv 0\,,$$

that is, either killing was present at some node, or the process is killed upon exiting U. This can also be seen by observing that if the killing rate was entirely zero on U, one would have that the Dirichlet energy is invariant under global shifts, i.e. for $f\in\mathbb{R}^U$, $\mathscr{E}_U(f,f) = \mathscr{E}_U(f+c,f+c)$, $c\equiv$constant, a clear contradiction to the centeredness of the GFF.

[1] For instance, taking balls in the graph or chemical distance of radius $n\geq 1$ centred about some point $x_0\in G$

2) This proposition can be used for a different construction of the canonical law $\mathbb{P}$, of the GFF. Indeed, one can take an exhaustion of G by an increasing sequence of finite (compact in the discrete topology) sets $(U_n)_{n\geq 1}$, $U_n \subseteq U_{n+1}$, $n \geq 1$ and $\cup_{n\geq 1} U_n = G$[1] , and **define** the GFF $(\varphi^n)_{x\in G}$ by setting it zero on $G \setminus U_n$ and on U_n define the GFF as the Gaussian vector on $\mathbb{R}^{U_n}$ using the density above. Taking $n \to \infty$, one can show that the canonical law of $(\varphi^n)_{x\in G}$, $\mathbb{P}^n \to \mathbb{P}$ weakly to a Gaussian field, which has the law of the GFF (This argument has the flavour of an argument by means of abstract machinery such as Kolmogorov extension theorem, which one typically uses to construct a stochastic process).

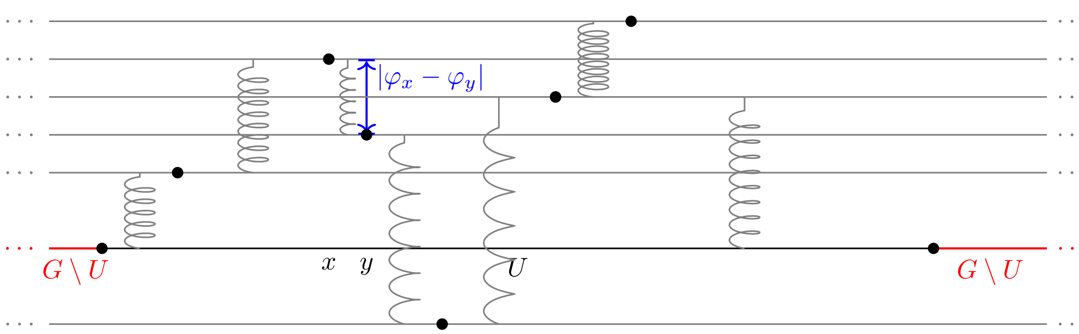
Proof. By construction, $(\varphi_x)_{x\in U}$ is a centred Gaussian vector with covariance matrix G_U given by the Green's function g_U. It thus suffices to show G_U is invertible and obtain an explicit form for its inverse.

By Proposition 2.14 and a simple computation, we have

$$\mathscr{E}_U(\varphi,\varphi) = (\varphi, -\Delta_U \varphi) = \sum_{x,y\in U} \varphi_x\varphi_y(1_x, -\Delta_U 1_y)\,.$$

Now, by Lemma 2.12, $-\Delta_U g^y = (\mu_y)^{-1} 1_y$, $y \in U$, which means $((1_x, -\Delta_U 1_y))_{x,y\in U}$ is the inverse to G_U, and the rest follows by a change of variables. □

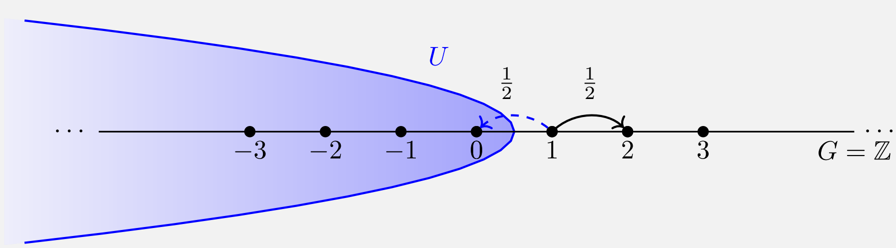
Proposition 3.4 gives some intuition as to the local behaviour of the GFF (see Figure 1 for a simulation of the GFF). The 'gradient' term in the Dirichlet energy associated to U, which can be interpreted as the sum total of potential energies associated to springs (with spring constants depending on the killing rates $\kappa^U_\cdot$ and capacitances $\mu^U_{\cdot,\cdot}$ with displacements $|\varphi_x - \varphi_y|$, $x, y \in U$ neighbouring points in the graph; hence, large relative displacements are penalised. In conjunction with $\varphi_{\partial(G\setminus U)} \equiv 0$ (from $g_U(x, y) = 0$, x or y not in U) gives rise to the following 'bedspring picture' for the GFF (the black dots correspond to the values of the GFF over U):



By means of a simple, yet important example, we see that the GFF is hiding many familiar structures.

Examples 3.5. Let $U \equiv (-\infty, 0] \cap \mathbb{Z}$ and define the weighted graph $T^0 = (G^0, \mu^0, \kappa^0)$ with

- $G^0 = \{1, 2, 3, \cdots\}$,
- $\mu^0 = \mu|_{G^0}$,
- $\kappa^0_x = \delta_{x,1}$, $x \in G^0$.



By Proposition 3.4, one can deduce that under $\mathbb{P}_U$,

$$\varphi_0 = 0\,, \varphi_1 = (\varphi_1 - \varphi_0) \sim \mathscr{N}(0, \underbrace{(\kappa_0^0)^{-1}}_{=1})\,, (\varphi_x - \varphi_{x-1})_{x\geq 2} \sim \mathscr{N}(0, \underbrace{(\mu^0_{x-1,x})^{-1}}_{=1})\,.$$

Indeed, this follows from a localisation argument by considering $G^0 \cap [0,n]$, $n \geq 1$ and showing that the density one obtains for the GFF on $G^0 \cap [0,n]$ $(\varphi^n_x)_{1\leq x\leq k}$, with $k \geq 1$ fixed, the density converges to that specified by the above. Then, by monotone convergence of the greens functions, one also obtains the weak convergence of the law of the GFF on $G^0 \cap [0,n]$ to $\mathbb{P}_U$ as $n \to \infty$. Hence, under $\mathbb{P}_U$,

$$\varphi_n = \varphi_n - \varphi_0 = \sum_{k=1}^{n} \underbrace{\xi_k}_{iid.\sim\mathscr{N}(0,1)}\,,$$

so $(\varphi_n)_{n\geq 0}$ is just a random walk with Gaussian increments.

Now, we can iterating the above construction, by refining the scale of the underlying graph. For $n \geq 1$, define

$$G^n \equiv 2^{-n}\mathbb{Z} \cap (0,\infty)\,, \mu^n_{x,x+2^{-n}} = 2^n\,, \kappa^n_x = 2^n\delta_{x,2^{-n}} x \in G^n\,.$$

Now, with $(B^n_t)_{\geq 0}$ the linear interpolation of $(\varphi^n_x)_{x\in G^n}$, the GFFs on G^n, we have by Donsker's theorem that $(B^n_t)_{\geq 0} \implies (B_t)_{t\geq 0}$ in distribution where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

For $\mathbb{Z}^d\,, d \geq 2$, the story is more complicated; by imitating the above procedure, one does indeed obtain a scaling limit, though not as a random function, but as a **random distribution**.

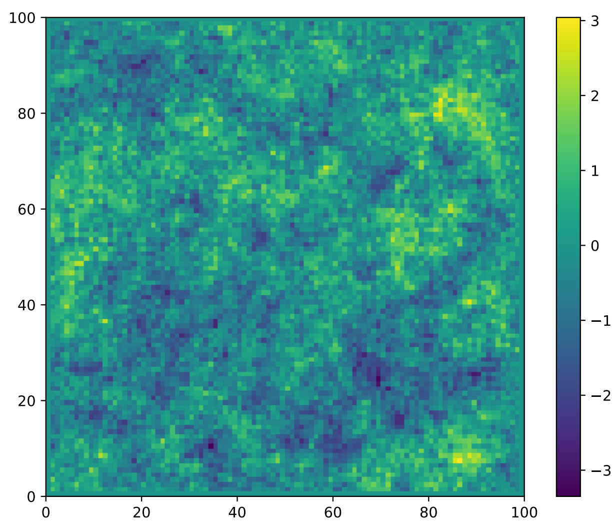


Figure 1: An illustration of the Gaussian Free field on the lattice $\mathbb{Z}^2$ (with natural weights) killed upon exiting the box $[0,100]^2$.

We now explore how the GFF in one region influences the GFF in another region. This will lead us to the notion of the (domain) Markov property enjoyed by the GFF on transient weighted graphs.

More precisely, fix $\Gamma = (G,\mu,\kappa)$ a weighted transient graph and let $\mathbb{P}$ denote the canonical law of the GFF on Γ. Fix $K \subseteq G$ with finite boundary, that is $|\partial K| < \infty$. We want to understand the effect on $(\varphi_x)_{x\in G\setminus K}$ upon conditioning by $(\varphi_x)_{x\in K}$. Since we are dealing with Gaussian fields, one would expect the conditioning to preserve the Gaussian structure of the field (as it does in the case of finite dimensional Gaussian vectors). In fact more is true, we can compute the mean and covariance structure of the conditioned field explicitly, in terms of the field corresponding to the graph with killing the random walk as soon as it exits $U \equiv G \setminus K$ and random walk probabilities.

Proposition 3.6. *Define the random (Gaussian) field h on G by setting*

$$h_x = \sum_{y \in K} P_x(H_K < \infty, X_{H_K} = y) \cdot \varphi_y \,,$$

which is well-defined since $|\partial K| < \infty$. Note that $h|_K = \varphi|_K$. Also define the Gaussian field $\psi. = \varphi. - h.$ (such that $\psi|_K = 0$). Then, under $\mathbb{P}$,

i) $\sigma(h_x : x \in G) \perp\!\!\!\perp \sigma(\varphi_x : x \in G)$[a],

ii) $\psi = (\psi_x)_{x \in G}$ *has law $\mathbb{P}_U$, with $U = G \setminus K$.*

[a] $\perp$ denotes independence of sigma algebras.

Proof. To prove i), it suffices to check $h., \psi.$ are uncorrelated, since all the fields in question are centred Gaussian fields. Now, we compute for all $x \in K, y \in U$

$$\begin{aligned}
\mathbb{E}[\varphi - x\psi_x] &= \mathbb{E}[\varphi_x(\varphi_y - h_y)] = \mathbb{E}[\varphi_x \varphi_y]\mathbb{E}[\varphi_y h_y] \\
&= g(x,y) - \sum_{\zeta \in K} P_y(H_K < \infty, = \zeta) \underbrace{\mathbb{E}[\varphi_x \varphi_\zeta]}_{=g(x,\zeta)} \\
&= g(x,y) - E_y[1(H_K < \infty) g(X_{H_K}, x)] \\
&= g_U(x,y) = 0 \,. && \text{(Lemma 2.4 and } x \notin U)
\end{aligned}$$

By linearity, we conclude $\mathbb{E}[h_x \psi_y] = 0$ for all $x \in G$, $y \in U$.

To prove ii), it suffices to show that $\mathbb{E}[\psi_x \psi_y] = g_U(x,y)$, $x, y \in G$. Observe this is immediate if either $x \notin U$, or $y \notin U$. So now fix $x, y \in U$. We now compute the covariance

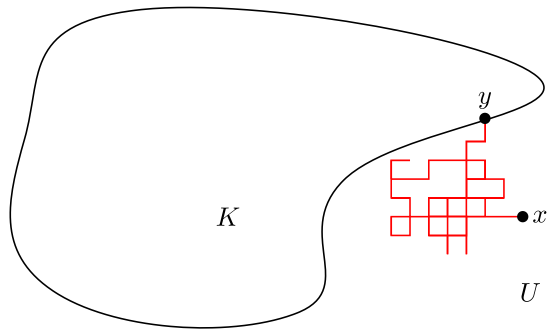
$$\begin{aligned}
\mathbb{E}[h_x h_y] &= \sum_{\zeta, \zeta' \in K} P_x(H_K < \infty, X_{H_K} = \zeta) \cdot P_y(H_K < \infty, X_{H_K} = \zeta') \cdot \underbrace{\mathbb{E}[\varphi_\zeta \varphi_{\zeta'}]}_{=g(\zeta,\zeta')} \\
&= \sum_{\zeta \in K} P_x(H_K < \infty, X_{H_K} = \zeta) \cdot (g(y,\zeta) - \underbrace{g_U(y,\zeta)}_{=0(\zeta \in K)}) && \text{(Lemma 2.4)} \\
&= g(x,y) - g_U(x,y) \,.
\end{aligned}$$

This now gives

$$\begin{aligned}
\mathbb{E}[\psi_x \psi_y] &= \mathbb{E}[(\varphi_x - h_x)(\varphi_y - h_y)] \\
&= g(x,y) - (\mathbb{E}[\varphi - x h_y] + \mathbb{E}[h_x \varphi_y] - \mathbb{E}[h_x h_y]) \\
&= g(x,y) - (2-1)(g(x,y) - g_U(x,y)) \\
&= g_U(x,y) \,.
\end{aligned}$$

□

Remark. The above result gives that conditionally on $(\varphi_x)_{x \in K}$, φ_x $x \in U$ is Gaussian with **mean** h_x and and **variance** $g_U(x,x)$. The expression for h_x, $x \in U$ is the average field location when the random walk started from x first hits ∂K (see the illustration below), and is also known as a harmonic average.



We now readily obtain the following corollary, which converts the statement in Proposition 3.6 in terms of conditional expectations.

Corollary 3.7. Fix $K \subsetneq G$, $|\partial K| < \infty$ and set $U = G \setminus K$. Then, for all $f : \mathbb{R}^G \to \mathbb{R}$ bounded measurable one has

$$\mathbb{E}[f(\varphi_\cdot)|\sigma(\varphi_x : x \in K)] = \tilde{E}_U[f(\tilde{\varphi}_\cdot + h_\cdot)]\,, \qquad \mathbb{P} - \text{a.s.}$$

Here $\tilde{E}$ denotes the expectation operator associated to an independent copy of the GFF $\tilde{\varphi}$.

Proof. To prove this, one needs to check that the RHS is indeed a regular version of the conditional expectation, i.e. the LHS. This can be done by a standard monotone-class argument. □

We now explore what these 'interaction' terms from the boundary $h_\cdot$ look like depending on the domain K in the lattice $\mathbb{Z}^d$, $d \geq 3$.

Examples 3.8. Consider $\mathbb{Z}^d$, $d \geq 3$ with natural weights, that is $\mu_\cdot \equiv 1$, $\kappa_\cdot \equiv 0$. Furthermore, fix $U \subseteq G$ finite with $x \in U$.

Observe that one always has

$$1 \leq g_U(x,x) \leq g(x,x)\,, \qquad x \in U\,,$$

which means that the GFF $\psi \sim \mathbb{P}_U$ has 'sizeable' fluctuations, no matter how small we make U. Moreover, we can compute at every $x \in U$, the variance

$$\mathrm{var}_{\mathbb{P}}(h_x) = \mathbb{E}[h_x^2] = g(x,x) - g_U(x,x) = E_x[g(x, X_{H_K}) \cdot 1(H_K < \infty)] \leq P_x(H_K < \infty) \cdot \sup_{\zeta \in \partial K} g(x,\zeta)\,.$$

Now, with $U = B(x,n)$, $n \geq 1$, a ball centred at $x \in \mathbb{Z}^d$ of radius n in $\mathbb{Z}^d$ with respect to the graph distance, say, we have by transience $P_x(H_K < \infty) = 1$ and the asymptotics $\sup_{\zeta \in \partial(G \setminus U)} g(x,\zeta) \asymp n^{2-d}$, $n \geq 1$.

In contrast, when one imposes a uniform lower bound on the killing rate, i.e. $\kappa_x \geq c > 0$ for all $x \in \mathbb{Z}^d$, one has the exponential decay, $P_x(H_K < \infty) \leq \mathrm{e}^{-cn}$, leading to a much smaller variance for h_x, $x \in U$ (it is very unlikely for the random walk to survive all the way from $x \in \mathbb{Z}^d$ to $\partial B(x,n)$ due to the presence of killing). If one tunes the killing lower bound $c \to \infty$, the above shows that the GFF converges to a field of iid. centred Gaussians.

In both cases, one sees that by taking $n \to \infty$, one recovers the original field.

4 Percolation for the GFF

Let $\gamma = (G, \mu, \kappa)$ be a weighted transient graph, with $|G| = \infty$. Let φ denote the GFF on Γ; we will use the varying parameter $a \in \mathbb{R}$ (analogous to the inverse temperature in the Ising, polymer models) and do percolation using the level sets of the GFF $\{\varphi \geq a\} = \{x \in G : \varphi_x \geq a\}$.

We now establish some notation. For $A, B \subseteq G$, define the event of connection

$$\{A \overset{\geq a}{\leftrightarrow} B\} \overset{\text{def}}{=} \{\exists \gamma \text{ path } , \text{range}(\gamma) \cap A \neq \emptyset \,, \text{range}(\gamma) \cap B \neq \emptyset \,, \text{range}(\gamma) \subseteq \{\varphi \geq a\}\} \,.$$

Note that $\overset{\geq a}{\leftrightarrow}$ is an equivalence relation on $\{\varphi \geq a\}$; connected components of $\{\varphi \geq a\}$ with respect to this equivalence relation are called **clusters**. Call he **percolation event**

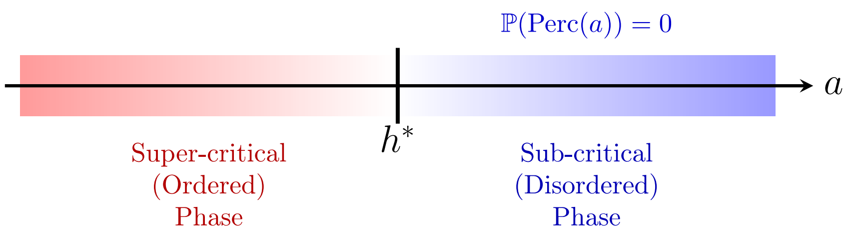
$$\text{Perc}(a) \overset{\text{def}}{=} \{\exists \text{ unbounded (infinite) cluster}\} = \bigcup_{x \in G} \underbrace{\bigcap_{n \geq 1} \{x \overset{\geq a}{\leftrightarrow} \partial B(x, n)\}}_{\equiv \{x \overset{\geq a}{\leftrightarrow} \infty\}} \,.$$

Now, a pertinent question is: are there infinite clusters? In other words, what is $\mathbb{P}(\text{Perc}(a))$, $a \in \mathbb{R}$? One can immediately observe that the events $\text{Perc}(a)$ are decreasing in a, hence so are the probabilities $\mathbb{P}(\text{Perc}(a))$. Moreover, by the continuity of probability measures, $\mathbb{P}(\text{Perc}(a))$ is left-continuous in $a \in \mathbb{R}$. Also, note that when the killing is effectively set to be infinite, the GFF corresponds to a field of iid centred Gaussians. Thus, one recovers Benoulli percolation $p = \mathbb{P}(\varphi_x \geq a)$.

Now, associated to the graph Γ, we define the critical parameter $h^* = h^*(\Gamma)$, defined as

$$h^* = \inf\{a \in \mathbb{R} : \mathbb{P}(\text{Perc}(a)) = 0\} \,.$$

We thus obtain the following qualitative picture.

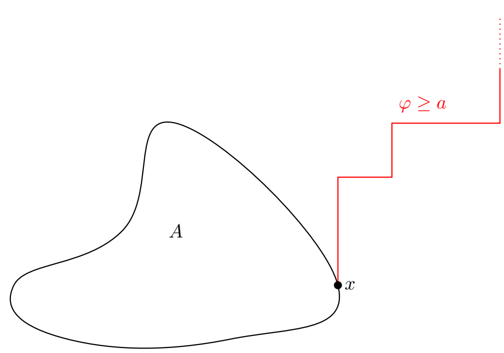


A priori, the critical parameter h^* can take the degenerate values $\pm\infty$, which corresponds to no phase transition. We are interested in investigating the conditions on Γ under which $|h^*(\Gamma)| < \infty$; in other words, when is there a phase transition?

4.1 Lower bounds on h^*

We now turn our focus to establishing lower bounds for the critical parameter h^*. For a finite set $A \subseteq G$, define the **disconnection event**

$$\mathcal{D}_A \overset{\text{def}}{=} \{A \overset{\geq a}{\not\leftrightarrow} \infty\} \left(= \bigcup_{n \geq 1} \{A \overset{\geq a}{\not\leftrightarrow} \partial B(x, n)\} \right) \,.$$



We have the following theorem.

Theorem 4.1. *Let Γ be a transient weighted graph with no killing, i.e. $\kappa_\cdot \equiv 0$. Then, for all $a < 0$,*

$$\mathbb{P}(\mathscr{D}_A) \leq \mathrm{e}^{-\frac{1}{2}a^2 \mathrm{cap}(A)}$$

Before we prove Theorem 4.1, we state and prove an easy corollary.

Corollary 4.2. Let Γ be a transient weighted graph with no killing. Then, the critical parameter satisfies

$$h^*(\gamma) \geq 0\,.$$

Proof. Fix any $x \in G$ and consider the singleton $A = \{x\}$. Then, by inclusion we have the bound

$$\begin{aligned}\mathbb{P}(\mathrm{perc}(a)) &\geq \mathbb{P}(x \overset{\geq a}{\leftrightarrow} \infty) \\ &= 1 - \mathbb{P}(\mathscr{D}_A) = 1 - \mathrm{e}^{-\frac{1}{2}a^2\mathrm{cap}(A)}\,. \qquad \text{(Theorem 4.1)}\end{aligned}$$

We know that $\mathrm{cap}(A) = 1/g(x,x) > 0$ by example 2.8 and transience. This gives

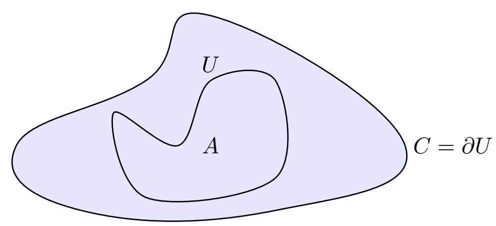
$$\mathbb{P}(\mathrm{Perc}(a)) \geq 1 - \mathrm{e}^{-\frac{1}{2}\frac{a^2}{g(x,x)}} > 0\,, \qquad a < 0\,.$$

This means for any $a < 0$, $h^*(\Gamma) \geq a$, and conclude by taking $a \nearrow 0$. □

Remark. When the killing rate has a uniform lower bound $\kappa_\cdot \geq c > 0$, one can show by adapting the arguments from Bernoulli percolation that $h^*(\Gamma) > -\infty$.

Before we start the proof of Theorem 4.1, we need some preliminary results of a topological flavour.
Fix $A \subseteq G$ finite and recall the definition of the boundary of A,

$$\partial A = \{y \in G \setminus A : y \sim x\,, x \in A\}\,.$$



We say $C \subseteq G$ is a **contour surrounding** A if there exists $G \supseteq U \supseteq A$, $|U| < \infty$ with $C = \partial U$. Note that C always determines U, and so we can unambiguously write $U = \mathrm{Int}C$. Now, for $n \geq 1$ and contours $C_1, \cdots, C_n$ surrounding A, we can define their 'maximum' as

$$\max\{C_1, \cdots, C_n\} \overset{\text{def}}{=} \partial\left(\bigcup_{k=1}^{n} \mathrm{Int}C_k\right)\,.$$

One readily sees that this is indeed a contour surrounding A.

We now have the following lemma that gives a characterisation of disconnection events in terms of contours.

Lemma 4.3. *Let $V \subseteq G$, $|V| < \infty$, $A \subseteq V$, and set the disconnection event*

$$\mathscr{D}_{A,V} \overset{\text{def}}{=} \{A \overset{\geq a}{\not\leftrightarrow} \partial V\}\,.$$

Then,

$$\mathscr{D}_{A,V} = \{\exists \text{ contour } C \text{ surrounding } A \text{ with } \mathrm{Int}C \subseteq V\,, \varphi_x < a, x \in C\}\,.$$

Proof. The inclusion $\supseteq$ is clear. To prove the inclusion $\subseteq$, one can take the set

$$U = A \cup \{x \in G : x \overset{\geq a}{\leftrightarrow} A\}\,.$$

Since $A \subseteq V$ and $A \overset{\geq a}{\not\leftrightarrow} \partial V$, it follows, $U \subseteq V$ and by definition, on $C \equiv \partial U$, we have $\varphi_x < a, x \in C$, concluding the proof. $\square$

We are now ready to prove Theorem 4.1.

(Proof of Theorem 4.1). Let V be a finite set with $A \subseteq V \subseteq G$ (which we will take to increase to G). Then, on the disconnection vent $\mathcal{D}_{A,V}$, since there are only finitely many contours C surrounding A inside V, we can define their maximal contour as

$$C^{\max} = \max\{ \text{ contour } C \text{ surrounding } A \text{ with } \mathrm{Int}C \subseteq V\,, \varphi_x < a, x \in C\}\,.$$

The key to observe is that $C^{\max}$ is a discrete, finitely supported random variable that satisfies the following measurability property for a contour C:

$$\{C^{\max} = C\} \in \sigma(\varphi_x : x \in G \setminus \mathrm{IntC})\,.$$

This is because, on a fixed contour C, to determine whether it is maximal or not, one only has to 'explore' the region in $G \setminus \mathrm{IntC}$ and the values of the GFF therein.

We can thus express the disconnection event as a disjoint union

$$\mathcal{D}_{A,V} = \bigsqcup_{C \text{ contour}} \{C^{\max} = C\}\,,$$

expressing the fact that the disconnection event is precisely the event that there exists a maximal contour.

We now estimate for any $a < 0$, $A \subseteq V \subseteq G$, $|V| < \infty$, with the field,

$$\begin{aligned}
\mathbb{P}(\langle \overline{e}_A, \varphi\rangle \leq a) &\geq \mathbb{P}(\langle \overline{e}_A, \varphi\rangle \leq a, \mathcal{D}_{A,V}) \\
&= \sum_{C \text{ contour}} \mathbb{P}(\langle \overline{e}_A, \varphi\rangle \leq a, C^{\max} = C) \\
&\geq \sum_{C \text{ contour}} \overbrace{\mathbb{P}(\langle \overline{e}_A, \psi\rangle \leq 0, \underbrace{\langle \overline{e}_A, h\rangle \leq a, C^{\max} = C}_{\in \sigma(\varphi_x : x \in G\setminus \mathrm{IntC})})}^{h.\text{ as defined in 3.6 with } K = G\setminus \mathrm{Int}C} \\
&= \sum_{C \text{ contour}} \underbrace{\mathbb{P}(\langle \overline{e}_A, \psi\rangle}_{\text{centred Gaussian}\leq 0)} \cdot \mathbb{P}(\langle \overline{e}_A, h\rangle \leq a, C^{\max} = C) \quad \text{(Proposition 3.6)} \\
&= \sum_{C \text{ contour}} \frac{1}{2}\mathbb{P}(\langle \overline{e}_A, h\rangle \leq a, C^{\max} = C)\,.
\end{aligned}$$

Now, observe that on the event $\{C^{\max} = C\}$ the fact that there is no killing gives

$$\langle \overline{e}_A, h\rangle = \frac{1}{\mathrm{cap}(A)} \sum_{x \in A, y \in C} e_A(x) \cdot P_x(T_{\mathrm{Int}C} < \infty, X_{T_{\mathrm{Int}C}} = y) \cdot \varphi_y \leq a\,,$$

so we estimate from below

$$\mathbb{P}(\langle \overline{e}_A, \varphi\rangle \leq a) \geq \sum_{C \text{ contour}} \frac{1}{2}\mathbb{P}(C^{\max} = C) = \frac{1}{2}\mathbb{P}(\mathcal{D}_{A,V})\,.$$

Now, one also computes the variance of the centred Gaussian $\langle \overline{e}_A, \varphi\rangle$,

$$\mathrm{var}(\langle \overline{e}_A, \varphi\rangle) = E(\overline{e}_A, \overline{e}_A) = \frac{1}{\mathrm{cap}(A)}\,,$$

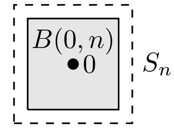
and to conclude use the tail bounds for $X \sim N(0, \sigma^2)$, $\sigma > 0$,

$$\mathbb{P}(X > |a|) \leq \frac{1}{2}\mathrm{e}^{-\frac{1}{2}\frac{a^2}{\mathrm{cap}(A)}}\,, \qquad a < 0\,,$$

take $V \nearrow G$ and use monotone convergence. $\square$

Remark. On $\mathbb{Z}^d$, $d \geq 3$ with natural weights, we saw that the disconnection probability of a ball of radius $n \geq 1$ from infinity is bounded by

$$\mathbb{P}(B(0,n) \overset{\geq a}{\not\leftrightarrow} \infty) \leq \mathrm{e}^{-ca^2 n^{d-2}}\,, n \geq 1\,, a < 0\,.$$



One can sow this is **indeed sharp** (up to first order in the exponent). One should contrast this with Bernoulli percolation where one obtains the sharp estimate with an exponent of n^{d-1}. This can be seen from covering $B(0,n)$ by a shell S_n which has $\mathscr{O}(n^{d-1})$ elements and use independence.

So one sees that these two percolation models are in different 'regimes'.

We now prove the following lemma for tail events for the GFF on a transient weighted graph.

Lemma 4.4. *Let $\Gamma = (G, \mu, \kappa)$ be a transient weighted graph. Then, for all $A \subseteq G$, $|A| < \infty$, and $a > 0$,*

$$\mathbb{P}(\varphi_x > a, x \in A) \leq \mathrm{e}^{-\frac{a^2}{2}\mathrm{cap}(A)}\,.$$

Proof. Let ν be any probability measure supported on A. Then, we have by inclusion,

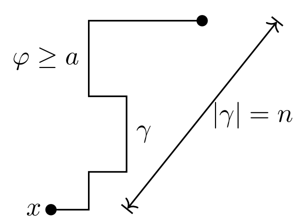
$$\mathbb{P}(\varphi_x > a, x \in A) \leq \mathbb{P}(\langle \nu, \varphi \rangle > a) \leq \mathrm{e}^{-\frac{a^2}{2E(\nu,\nu)}}\,,$$

where the second inequality follows from the fact that $\langle \nu, \varphi \rangle$ is a centred Gaussian with variance $E(\nu,\nu)$. Now, optimising over ν and using the variational characterisation for capacity, Theorem 2.18, we conclude. $\square$

We now prove the following quick corollary giving the existence of a super-critical regime, assuming a uniform lower bound on the killing rate and G being of bounded degree. The key here is that the killing gives strong control on the capacity of arbitrary sets subsets of G.

Corollary 4.5. If Γ has bounded degree and the killing rate satisfies the uniform lower bound $\inf_{x\in G} \kappa_x > c > 0$ for some $c > 0$, then $h^*(\Gamma) < \infty$.

Proof. For $x \in G$, let $N(x)$ be the degree of x in G, so that $\sup_{x\in G} N(x) < \infty$. It is not hard to see that the number of self-avoiding paths γ from x of length $n \geq 1$ is estimated by $N(x) \cdot (N(x)-1)^{n-1} \leq c_0^n$ for some constant $c_0 > 0$. We thus estimate by a union bound,



$$\mathbb{P}(x \overset{\geq a}{\leftrightarrow} \partial B(x,n)) \leq c_0^n \sup_\gamma \mathbb{P}(\varphi_x > a, x \in \mathrm{Range}(\gamma))\,.$$

Now by Lemma 4.4, this is further estimated by $c_0^n \mathrm{e}^{-\frac{a^2}{2}\mathrm{cap}(\mathrm{Range}(\gamma))}$. The goal is to show this is $o(1)$ as $n \to \infty$ for potentially $a \gg 1$. Observe that due to killing, we have for all $A \subseteq G$, the lower bound on the equilibrium measure

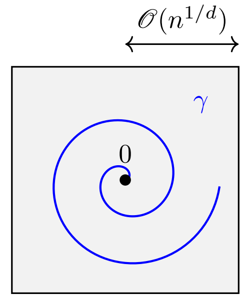
$$\overline{e}_A(x) = \mu_x P_x(\tilde{H}_x = \infty) \geq \mu_x \cdot \overbrace{\frac{\kappa_x}{\mu_x}}^{\text{prob. to die at first jump}} = \kappa_x \geq \inf_{x\in G} \kappa_x > c > 0\,, \qquad x \in A\,.$$

This gives $\mathrm{cap}(A) \geq c|A|$. Now, with $A = \gamma$, we obtain the lower bound $\mathrm{cap}(\mathrm{Range}(\gamma)) \geq c|\gamma| = cn$, so

$$\mathbb{P}(x \overset{\geq a}{\leftrightarrow} \partial B(x,n)) \leq c_0^n \mathrm{e}^{-\frac{a^2}{2}cn} \overset{n\to\infty}{\longrightarrow} 0\,,$$

for $a > (\log^{\frac{1}{2}} c)/(2c_0)^{\frac{1}{2}} > 0$, completing the proof. $\square$

We illustrate by example why this approach does not generalise to other Γ's, for example, with no killing. The difficulty is to control the capacity of rather arbitrary paths, which can be 'very windy' or 'bunched up'. For instance, consider the case $\mathrm{Range}(\gamma) \subseteq B(0, cn^{\frac{1}{d}})$, $c > 0$ and $|\gamma| = n$. Hence, we have by the variational characterisation for capacity, Theorem 2.18,



$$\mathrm{cap}(\mathrm{Range}(\gamma)) \le \mathrm{cap}(B(0, cn^{\frac{1}{d}})) \asymp n^{\frac{d-2}{d}} .$$

Another case would be to take

$$\mathrm{diam}(\mathrm{Range}(\gamma) \ge cn ,$$

for $c > 0$ constant. Again, using variational principles, one has examples where $\mathrm{cap}(\mathrm{Range}(\gamma)) = o(n)$.

In both cases, the capacity of the image of the path γ grows sub-linearly, which is not enough to make the argument in Corollary, 4.5 work. One might object that the bound coming from lemma 4.3 is not sharp; however, in practice, one is typically not far from it (for instance in $\mathbb{Z}^d$). For instance, in $\mathbb{Z}^d$, $d \ge 3$, we have the estimate for $a > 0$,

$$\underbrace{\mathrm{e}^{-ca^2 n^{d-2} \log n}}_{\text{sharp}} \le \mathbb{P}(\varphi_x \ge a, x \in B(0, n)) \overset{\text{Lemma4.3}}{\le} \mathrm{e}^{-c_1 a^2 n^{d-2}} .$$

Again, notice that for Bernoulli percolation, one has the sharp asymptotic $\asymp \mathrm{e}^{-n^d}$, which is readily available by independence.

We now briefly give an example where the critical parameter is can be determined exactly, and is in fact $h^*(\Gamma) = 0$.

Examples 4.6. Consider $\Gamma = (\mathbb{Z}, \mu, \kappa \equiv 0)$, with weights $\mu_{k,k+1} = \alpha^k$, $k \in \mathbb{Z}$ for some $\alpha > 1$. By the Markov property, we have that the random walk X on Γ can be expressed P_x-almost surely as

$$X_n \overset{\text{law}}{=} x + \sum_{k=1}^{n} \xi_k \, , \ge 1$$

with the ξ_k, $k \ge 1$ being iid Bernoulli random variables taking values in $\{-1, 1\}$ having success probability $\alpha/(1+\alpha)$. By the strong law of large numbers, we have almost surely

$$\lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}[\xi_1] = \frac{\alpha - 1}{\alpha + 1} > 0 .$$

Hence, we see that X is a transient random walk with upward drift, and in particular, $P_x(\lim_{n\to\infty} X_n = \infty) = 1$ for all $x \in \mathbb{Z}$.

One can obtain the functional form of the Green's function on Γ explicitly.

Lemma 4.7. *For all $n, k \in \mathbb{Z}$, $n \ge k$, $g(n, k) = g(n, n) = \alpha^{-n} g(0, 0)$.*

Proof. Let $U_n = (-\infty, n) \cap \mathbb{Z}$. Then by Lemma 2.4, we have

$$g(n, k) = \overbrace{g_{U_n}(n, k)}^{=0} + \overbrace{P_k(H_n < \infty)}^{=1} g(n, n).$$

For the second equality, observe that the hitting probabilities in Definition 2.2 are homogeneous in $n \in \mathbb{Z}$, and so one obtains the desired result upon renormalising by the conductance $\mu_n = \mu_{n,n-1} + \mu_{n,n+1} = \alpha^{n-1} + \alpha^{n+1}$, noting that $\mu_0 = \alpha^{-1} + \alpha$. □

Now, on $U_0 = (-\infty, 0] \cap \mathbb{Z}$, we can express by Lemma 2.4 the Green's functions on Γ_{U_0}, the graph

Γ with killing upon hitting U_0,

$$g_{U_0}(n,n) = g(n,n) - P_n(H_0 < \infty) g(0,n) = g(n,n)(1 - P_n(H_0 < \infty))$$
$$\overset{(2.5)}{=} g(n,n)\left(1 - \frac{g(0,n)}{g(0,0)}\right) = \alpha^{-n} g(0,0)(1-\alpha^{-n}), \qquad n \geq 1,$$

where the second equality follows from the lemma above. We thus have by the monotonicity of capacity (from the variational principles)

$$\mathrm{cap}_{U_0}([k,n]) \overset{\text{this is 'almost' an equality by transience for } n \gg 1}{\geq} \mathrm{cap}_{U_0}(\{n\}) = \frac{1}{g_{U_0}(n,n)}$$
$$= \frac{\alpha^n}{g(0,0)(1-\alpha^{-n})} \overset{n \gg 1}{\gtrsim} \alpha^n \,, 0 \leq k \leq n\,.$$

We thus have by Lemma 4.4,

$$\mathbb{P}_{U_0}(\varphi_x > \varepsilon, x \in [k,n] \cap \mathbb{Z}) \lesssim \mathrm{e}^{-\frac{\varepsilon^2}{2}\frac{\alpha^n}{g(0,0)}} \to 0, \quad n \to \infty\,.$$

Hence, putting all of the above together, that $h^*(\Gamma_{U_0}) \leq \varepsilon$ for all $\varepsilon > 0$, now conclude by taking $\varepsilon \searrow 0$. The matching lower bound $h^*(\Gamma_{U_0}) \geq 0$ can be obtained by adapting the arguments in Corollary 4.2, or by observing that the variances of the centred Gaussians φ_x, $x \geq 1$ decay geometrically, and are summable. Thus, by Borel-Cantelli, we have that almost surely, $\varphi_n \to 0$ $n \to \infty$, this means that almost surely eventually the field is going to localise above any threshold $-\varepsilon$ for $\varepsilon > 0$ (this argument also gives the upper bound).

4.2 Multi-scale analysis

Recall that in order to prove the upper bound on the critical parameter $h^*(\gamma)$, it suffices to show

$$\mathbb{P}(x \overset{\geq a}{\leftrightarrow} \partial B(x,n)) \overset{n\to\infty}{\longrightarrow} 0, \quad a \gg 1\,.$$

We now (and in the remainder of this section) impose some uniform structural constraints on the local geometry of the weighted graph Γ, which we hope allow us to prove the above dispensing with the uniform lower bound on the killing rate.

Definition 4.8. *Let $\gamma = (G, \mu, \kappa)$ be a transient weighted graph. Then, we say, some for constants $c, c', c'' \in (0,\infty)$, $0 < \beta < \alpha < \infty$, that Γ*

- *has **controlled weights** (CW) if*

$$\mu_{x,y} \geq c\mu_x\,, \qquad \textit{for all } x \sim y \in G\,;$$

- *satisfies the **Volume growth** condition (V_α) if*

$$c'(n^\alpha \vee 1) \leq \mu(B(x,n)) \leq c''(n^\alpha \vee 1)\,, \qquad \textit{for all } x \in G, n \geq 1\,;$$

- *satisfies the **Green's function** condition (G_β) if*

$$g(x,y) \leq c(d(x,y)^{\beta-\alpha} \wedge 1)\,, \qquad \textit{for all } x,y \in G\,.$$

Remark. Our running example is $\gamma = (\mathbb{Z}^d, \mu \equiv 1, \kappa \equiv 0)$, which we have thus not been able give an upper bound on the critical parameter, and can be seen to satisfy all the above conditions with $\alpha = d, \beta = 2$ (here $d \geq 3$).

We now prove a quick lemma that allows us to find 'approximate lattices of scale $L \geq 1$' in our graph G, that will enable use to carry out the multi-scale analysis.

Lemma 4.9 (Approximate lattices)**.** *If the volume growth condition (V_α) holds, then for all $L \geq 2$, there exists $\Lambda_L \subseteq G$ such that*

$$\begin{cases} \bigcup_{x \in \Lambda_L} B(x, L) = G \\ d(x, y) \geq L\,, \qquad \text{for all } x, y \in \Lambda_L \\ |\Lambda_L \cap B(x, NL)| \leq C_1 N^\alpha\,, \qquad \text{for all } x \in G, N \geq 1\,. \end{cases}$$

Remark. On $\mathbb{Z}^d$, one can take the lattice $\Lambda_L \equiv (L+1)\mathbb{Z}^d$, i.e. the renormalised lattice at scale L.

Proof. Fix an enumeration of G, which can be done since it is at most countably infinite. Now, we will construct $\Lambda_L = \{x_1, x_2, \cdots\}$ as follows. Pick $x_1 \equiv x \in G$ and suppose $x_1, \cdots, x_k$ has been defined for $k \geq 1$; then pick x_{k+1} to be the minimal (with respect to the ordering induced by the enumeration of G) element **not** in $\bigcup_{\ell=1}^k B(x_\ell, L)$.

Now, by construction, one quickly sees the first two properties in the statement are satisfied. To prove the third, we need to use the volume growth condition which gives uniform control on the cardinality of balls (for possibly different μ-dependent constants $c, c' \in (0, \infty)$),

$$c(n^\alpha \vee 1) \leq |B(x, n)| \leq c'(n^\alpha \vee 1)\,, \qquad \text{for all } x \in G, n \geq 1\,.$$

Now, by the second condition, we can obtain a **disjoint** cover of $\Lambda_L \cap B(x, NL)$ of balls of radius $L/2$, one for each point in Λ_L. Then, one obtains by inclusion (constants can change from line to line)

$$\bigcup_{y \in \Lambda_L \cap B(x, NL)} B(y, L/2) \subseteq B(x, (N+1/2)L)\,,$$

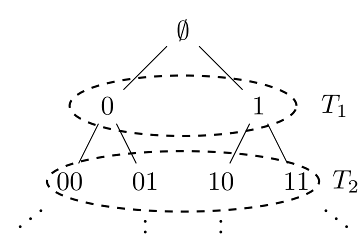
and by disjointness

$$cL^\alpha |\Lambda_L \cap B(x, NL)| \leq \inf_{y \in \Lambda_L \cap B(x, NL)} |B(y, L/2)| \cdot |\Lambda_L \cap B(x, NL)| \leq |B(x, (N+1/2)L)| \leq c' N^\alpha L^\alpha\,,$$

whence the result follows. □

We now make a small detour and set up some notation about trees. More precisely, let $T_0 = \{\emptyset\}$ be 'the tree with no leaves' and for $n \geq 1$ let $T_n = \{0,1\}^n$ denote the leaves of the binary tree of depth n, $\mathbb{T}_n = \cup_{k=1}^n T_k$. Now, for a node $m \in T_k$, $k \geq 1$, which can be expressed $m = (\xi_1, \cdots, \xi_k)$, for some binary sequence $\xi_1, \cdots, \xi_k \in \{0,1\}$ and $i \in \{0,1\}$, write

$$m_i = (\xi_1, \cdots, \xi_k, i) \in T_{k+1}\,,$$



which is also called the 'child' of m.

Now, we pick scales $L_n = 8^n$, $n \geq 1$ and for ease of notation set $\Lambda_n \equiv \Lambda_{L_n}$ for our approximate lattices of scale L_n. We now come to a notion of an 'embedding' of a tree into the graph G, that is 'well-separated' across the scales given by L_n, $n \geq 1$.

Definition 4.10 (Proper embedding)**.** *Fix $n \geq 1$. A map $\tau : T_n \to G$ is called a* ***proper embedding rooted at*** *$x \in \Lambda_n$ if*

- $\tau(\emptyset) = x$,
- $\tau(m) \in \Lambda_{n-k}$ *for all* $m \in T_k$, $0 \leq k \leq n$,
- $\tau(m_0) \in B(\tau(m), L_{n-k} + L_{n-k-1})$ *and*

$$3L_{n-k} - L_{n-k-1} \leq d(\tau(m_1), \tau(m)) < 3L_{n-k} + L_{n-k-1}$$

  for all $m \in T_k$, $0 \leq k < n$.

Moreover, for $n \geq 1, x \in G$, set

$$\mathcal{T}_{n,x} \stackrel{\text{def}}{=} \{\tau : \text{ proper embedding rooted at } x \in \Lambda_n\} .$$

We give an estimate for the cardinality of the number of proper embeddings which follows from the recursive definition thereof.

Lemma 4.11. *Fix any $x \in G$ and $n \geq 1$ and suppose the conditions in Definition 4.8 hold. Then, there exists some constant $C_2 > 0$ such that*

$$|\mathcal{T}_{n,x}| \leq C_2^{2^n} .$$

Proof. We proceed by induction. When $n = 0$, we have $|\mathcal{T}_{0,x}| = 1$ for all $x \in G$. Now, suppose the suppose is true for some $n - 1 \geq 1$ and all $x \in G$. Then, we have for every embedding τ in $\mathcal{T}_{n,x}$ that $\tau(\{0\}), \tau(\{1\}) \in B(x, 3L_n) \cap \Lambda_{n-1}$ and each of their descendants becomes a proper embedding rooted at $\tau(\{0\}), \tau(\{1\}) \in \Lambda_{n-1}$, hence we estimate

$$|\mathcal{T}_{n,x}| \leq |B(x, 3L_n) \cap \Lambda_{n-1}| \cdot \sup_{y \in B(x,3L_n) \cap \Lambda_{n-1}} |\mathcal{T}_{n-1,y}|^2 \stackrel{\text{Lemma4.9}}{\leq} (C_1 24^\alpha)^2 \cdot \sup_{y \in B(x,3L_n) \cap \Lambda_{n-1}} |\mathcal{T}_{n-1,y}|^2 ,$$

concluding the proof with $C_2 = (C_1 24)^\alpha$. ☐

We now prove that for any curve that 'extends over' a particular scale L_n, $n \geq 1$, one can find a sub-collection of points on the graph G, which are within unit distance to the image of the curve and are 'well-separated' across all smaller scales. In particular, these points will be the root nodes of a proper embedding and resemble 'Cantor dust' at scale L_n.

Lemma 4.12. *Suppose assumptions in Definition 4.8 hold. Fix $n \geq 0$, $x \in \Lambda_n$ and suppose $\gamma : \{0, 1, \cdots, |\gamma|\} \to G$ is a nearest neighbour path such that*

$$\text{im}(\gamma) \cap B(x, L_n) \neq \emptyset \neq \text{im}(\gamma) \cap B(x, 3L_n) ,$$

then there exists a proper embedding rooted at x, $\tau_{n,x}$ such that for all $m \in T_n$,

$$\text{im}(\gamma) \cap (B(\tau(m), \overset{=1}{L_0}) \neq \emptyset .$$

Proof. We will construct the embedding $\tau \in \mathcal{T}_{n,x}$ by inducting on $n \geq 0$.

First set $\tau(\emptyset) = x$. If $n = 0$, then we are done.

Now, For $n \geq 1$, cover $B(x, L_n)$ by smaller balls, $B(\xi, L_{n-1})$, for $\xi \in \Lambda_{n-1}$ (can be done by by Lemma 4.9). Now, γ has to hit some $B(\xi', L_{n-1})$, for $\xi' \in \Lambda_{n-1}$. Take $\tau(0) = \xi'$. Similarly, cover $\partial B(x, 3L_n)$ by smaller balls (again, by Lemma 4.9), $B(\xi, L_{n-1})$, for $\xi \in \Lambda_{n-1}$. Again, γ has to hit some $B(\xi'', L_{n-1})$, for $\xi'' \in \Lambda_{n-1}$. Take $\tau(1) = \xi''$.

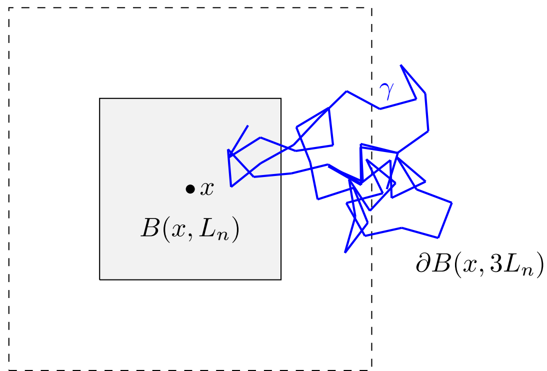
Now, observe that $\tau(0) \in B(\overset{=x}{\tau(\emptyset)}, L_n + L_{n-1})$ and $3L_n - L_{n-1} \le d(\tau(1), \overset{=x}{\tau(\emptyset)}) \le 3L_n + L_{n-1} \le 3L_n$. Moreover, by the above and the fact that γ is a nearest neighbour path,

$$\mathrm{im}(\gamma) \cap B(\tau(0), L_{n-1}) \neq \emptyset \neq \mathrm{im}(\gamma) \cap B(\tau(0), 3L_{n-1})\,,$$

and

$$\mathrm{im}(\gamma) \cap B(\tau(1), L_{n-1}) \neq \emptyset \neq \mathrm{im}(\gamma) \cap B(\tau(1), 3L_{n-1})\,.$$

Now by the induction hypothesis one takes the leaves of the proper embeddings associated to $\tau(0)$ and $\tau(1)$, which can be seen to satisfy all the requisite properties.



□

We now begin the proof of the existence of a non-trivial phase transition on a graph Γ satisfying the conditions in Definition 4.8 (recalling for transient Γ with no killing $h^*(\Gamma) \ge 0$, from Crollary 4.2.

Theorem 4.13. *Let Γ be a weighted transient graph satisfying the assumptions in Definition 4.8. Then, $h^*(\Gamma) < \infty$.*

Proof. By inclusion, it is enough to obtain the uniform decay

$$\sup_{x \in \Lambda_n} \mathbb{P}(B(x, L_n) \overset{\ge a}{\leftrightarrow} \partial B(x, 3L_n)) \overset{n\to\infty}{\longrightarrow} 0\,.$$

Note on this event, there exists a nearest-neighbour path γ meeting the conditions in lemma 4.12 such that for all $x \in \mathrm{im}(\gamma)$, $\varphi_x \ge a$. Moreover, by lemma 4.12, there exists a proper embedding $\tau \in \mathcal{T}_{n,x}$ such that for all leaves $m \in T_n$, there exists $x_m \in \mathrm{im}(\gamma)$ such that $x_m \in d(\tau(m), 1)$ and $\varphi_{x_m} \ge a$.

Now, associated to this embedding, let

$$A_\tau \overset{\text{def}}{=} \{x_m \in \mathrm{im}(\gamma) : m \in T_n\,, x_m \in d(\tau(m), 1)\}\,.$$

We now estimate for any $x \in \Lambda_n$,

$$\begin{aligned}
\mathbb{P}(B(x, L_n) \overset{\ge a}{\leftrightarrow} \partial B(x, 3L_n)) &\le \mathbb{P}\left(\bigcup_{\tau \in \mathcal{T}_{n,x}} \bigcup_{A_\tau} \{\varphi_x \ge a\,, x \in A_\tau\right) \\
&\le \sum_{\tau \in \mathcal{T}_{n,x}} |\mathcal{T}_{n,x}| \sup_{A_\tau} \mathbb{P}\left(\{\varphi_x \ge a\,, x \in A_\tau\}\right) && \text{(Union bound)} \\
&\le \sum_{\tau \in \mathcal{T}_{n,x}} C^{2^n} \sup_{A_\tau} \mathbb{P}\left(\{\varphi_x \ge a\,, x \in A_\tau\right) && \text{(Lemma 4.11)} \\
&\le \tilde{C}^{2^n} \sup_{A_\tau} \mathrm{e}^{-\frac{a^2}{2}\mathrm{cap}(A_\tau)}\,. && \text{(Lemma 4.4)}
\end{aligned}$$

It thus suffices to show that for any proper embedding $\tau \in \mathscr{T}_{n,x}$,

$$\operatorname{cap}(A_\tau) \geq C \cdot \overbrace{2^n}^{=|A_\tau|}, \quad n \geq 1,$$

for some constant $c > 0$. Note that this lower bound would be up to constants tight, as one can estimate the capacity a finite set by its cardinality.

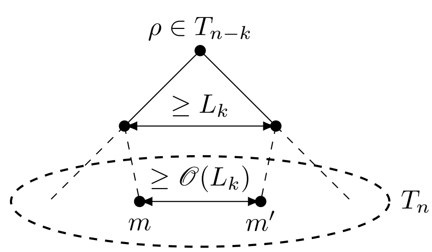
Now, for a leaf $m \in T_n$, and $0 \leq k \leq n$, define the sets

$$T_n^k(m) \overset{\text{def}}{=} \{m' \in T_n : \rho(m, m') \in T_{n-k}\},$$

that is the set of all leaves who have common ancestor with m in level $n-k$, which we denote $\rho(m,m')$ for two leaves $m, m' \in T_n$. Note, we can estimate the cardinalities for $0 \leq k \leq n$, $\sup_{m \in T_n} |T_n^k(m)| \leq 2^k$ as the first common ancestor is uniquely determined by m (the first k digits in the binary sequence defining m), and so all $m' \in T_n^k(m)$ must be in the binary tree with root $\rho(m,m')$ of depth $n-k$. Also note we can cover T_n by taking the union of all $T_n^k(m)$, $0 \leq k \leq n$ for all $m \in T_n$. We now estimate the capacity of A_τ using the variation principle in Theorem 2.18, by testing with the uniform measure $\nu = |A_\tau| 1_{A_\tau}$. We thus obtain the lower bound for the 'energy' of ν,

$$\begin{aligned} E(\nu,\nu) &= |A_\tau|^{-2} \sum_{m,m' \in T_n} g(x_m, x_{m'}) \\ &\leq |A_\tau|^{-1} \sup_{m \in T_n} \sum_{k=0}^{n} \sum_{m' \in T_n^k(m)} g(x_m, x_{m'}). \end{aligned}$$

Condition (G_α) on the Green's function in terms of the distance means it suffices to estimate the distances between $x_m, x_{m'}$, which we hope to achieve reasonable enough lower bounds, as they are within unit distance of leaves of a proper embedding, which we have ensured is 'well-separated' across all scales. We thus have by construction and repeated uses of the triangle inequality, see Definition 4.10, with $\rho \equiv \rho(x_m, x_{m'})$, for all $m \in T_n$, $m' \in T_n^k(m)$, $0 \leq k \leq n$,



$$\begin{aligned} d(x_m, x_{m'}) &\geq d(\tau(\rho_0), \tau(\rho_1)) - \sum_{\ell=1}^{k} (L_\ell + L_{\ell-1}) - 2L_0 \\ &\geq 3L_k - L_{k-1} - \sum_{\ell=1}^{k} (L_\ell + L_{\ell-1}) - 2L_0 \\ &\geq cL_k, \end{aligned}$$

for some universal constant $c > 0$. We have now,

$$\begin{aligned} E(\nu,\nu) &\leq |A_\tau|^{-1} \sup_{m \in T_n} \sum_{k=0}^{n} \sum_{m' \in T_n^k(m)} g(x_m, x_{m'}) \\ &\leq |A_\tau|^{-1} \sup_{m \in T_n} \sum_{k=0}^{n} \sum_{m' \in T_n^k(m)} d(x_m, x_{m'})^{\beta-\alpha} && \text{(Conditions 4.8)} \\ &\leq |A_\tau|^{-1} \sup_{m \in T_n} \sum_{k=0}^{n} |T_n^k(m)| \sup_{m' \in T_n^k(m)} d(x_m, x_{m'})^{\beta-\alpha} \\ &\leq |A_\tau|^{-1} \sup_{m \in T_n} \sum_{k=0}^{n} |T_n^k(m)| (cL_k)^{\beta-\alpha} \\ &\leq |A_\tau|^{-1} \sum_{k=0}^{n} 2^k (cL_k)^{\beta-\alpha} \leq |A_\tau|^{-1} \sum_{k=0}^{\infty} 2^k (cL_k)^{\beta-\alpha} \leq c|A_\tau^{-1}| = c2^{-n}, \end{aligned}$$

for $\beta < \alpha - \frac{1}{3}$. Though for general $\beta < \alpha$, one simply fine-tunes the scales $L_n = (f(\alpha, \beta))^n$, for some $f(\alpha, \beta) > 0$ (as well as the proper embedding definition and Lemma 4.12), to and follows through in the same manner. $\square$

To recap, so far, we have, under suitable constraints on the geometry of Γ, the very strong asymptotics for any $x \in G$,

$$\Theta_L^x(a) \overset{\text{def}}{=} \mathbb{P}(x \overset{\geq a}{\leftrightarrow} \partial B(x, L)) \leq c\mathrm{e}^{-L^{c'}} \overset{L\to\infty}{\longrightarrow} 0\,, a \gg 1\,,$$

for constants $c, c' > 0$. Note if the graph G is transitive (with uniform weights say), then one can drop the x-dependence in Θ. Moreover, by definition, we also have the convergence

$$\Theta_L^x(a) \overset{L\to\infty}{\longrightarrow} 0\,, a > h^*(\Gamma)\,.$$

Now, in the Physics literature, there are various predictions for Θ of the form

$$\Theta_L^x(a) \leq CL^{-\frac{1}{\rho}} f\left(\frac{L}{\xi(a)}\right)\,,$$

for some 'rapidly decaying' $f \in L^1(\mathbb{R})$, $\xi(a) = |a - h^*(\Gamma)|^{-\nu}$ and critical exponents $\nu, \rho > 0$ which are claimed to be 'universal' among certain classes of models. These estimates for Θ for a arbitrarily close to the critical parameter are known as 'non-perturbative', as opposed to the 'perturbative' bounds we have established, away from the critical parameter.

To study what happens near the critical phase as the distance $L \to \infty$, and hopefully establish estimates informed by the physics literature, one needs to study finer properties of the function Θ, which we do not pursue here.

References

[Dur19] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[Nor98] James R Norris. *Markov chains*. Number 2. Cambridge university press, 1998.